The inverse Willmore flow

Martin Mayer
diploma thesis
Mathematisches Institut der Universität Tübingen
Prof. Dr. Reiner Schätzle
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Abstract

Instead of investigating the Willmore flow for two-dimensional, closed immersed surfaces directly we turn to its inversion. We give a lower bound on the lifespan of this inverse Willmore flow, depending on the concentration of curvature in space and the extension of the initial surface, as well as a characterization of its breakdown, which might occur in finite time.

1 Introduction

The Willmore functional of a closed, immersed and smooth surface $f_0 : \Sigma \longrightarrow \mathbb{R}^n$ with induced area measure $d\mu = d\mu_{\Sigma,f_0}$ is given by

$$W(f_0) := \frac{1}{4}\int_\Sigma |H_{f_0}|^2 d\mu,$$

where $H_{f_0} = g^{i,j} A_{i,j}$ denotes the mean curvature. Then the Willmore flow is the L^2-gradient flow of the Willmore functional, i.e. a solution

$$f : \Sigma \times [0,T) \to \mathbb{R}^n \quad \text{with} \quad f(\cdot,0) = f_0 \quad \text{of}$$
$$\partial_t f = -\mathrm{grad}_{L^2} W(f) = -\frac{1}{2}(\Delta H + Q(A^0)H)$$

for some $T \in (0,\infty]$ and an arbitrary smooth initial surface f_0, see below.
If additionally $f : \Sigma \times [0,T) \longrightarrow \mathbb{R}^n \setminus \{\mathbb{0}\}$, we may consider the inverse flow

$$I_\sharp f := \frac{f}{\|f\|^2}.$$

Due to the fact, that the Willmore functional for surfaces is invariant under inversion,

$$W(f) = W(I_\sharp f),$$

cf. proposition 13.6, the inverse flow satisfies

$$\partial_t f = -\frac{\|f\|^8}{2}(\Delta H + Q(A^0)H).$$

This suggests to apply the techniques used by E. Kuwert and R. Schätzle [1], which we strongly recommend to read beforehand, since therein all the basic ideas and methods are performed with less calculations.
As our main result we give a lower bound on its lifespan, which depends on the concentration of curvature and the extension of the initial surface.

Theorem 1.1. *For $0 < R, \alpha < \infty$ and $n \in \mathbb{N}$ there exist*

$$\delta = \delta(n),\ k = k(n),\ c = c(n, R, \alpha) > 0$$

such that, if $\quad f_0 : \Sigma \longrightarrow \mathbb{R}^n \setminus \{\mathbb{0}\}\quad$ *is a closed immersed surface satisfying*

$$\sup_{x \in \mathbb{R}^n} \int_{B_\rho(x)} |A|^2 d\mu \le \delta,$$

$$\sup_{x \in \mathbb{R}^n} \rho^4 \int_{B_\rho(x)} |\nabla^2 A|^2 d\mu \le \alpha,$$

$$\rho^{-1} \|f_0\|_{L^\infty_\mu(\Sigma)} \le R$$

for some $\rho > 0$, there exists an inverse Willmore flow

$$f : \Sigma \times [0, T) \longrightarrow \mathbb{R}^n \setminus \{\mathbb{0}\}, \quad f(\cdot, 0) = f_0$$

with $T > \rho^{-4} c$. Moreover we have the estimates

$$\sup_{0 \le t \le \rho^{-4} c} \rho^{-1} \|f\|_{L^\infty_\mu(\Sigma)} \le 2R \quad \textit{and} \quad \sup_{0 \le t \le \rho^{-4} c,\, x \in \mathbb{R}^n} \int_{B_\rho(x)} |A|^2 d\mu \le k\delta.$$

Theorem 1.2. *For $n \in \mathbb{N}$ there exist*

$$\delta = \delta(n),\ k = k(n),\ c = c(n) > 0$$

such that, if $\quad f : \Sigma \longrightarrow \mathbb{R}^n \setminus \{\mathbb{0}\}\quad$ *is a closed immersed surface,*

$$T_{f_0, \mathbb{0}} > 0$$

maximal with respect to the existence of an inverse Willmore flow

$$f : \Sigma \times [0, T) \longrightarrow \mathbb{R}^n \setminus \{\mathbb{0}\}, \quad f(\cdot, 0) = f_0$$

and

$$\sup_{x \in \mathbb{R}^n} \int_{B_\rho(x)} |A|^2 d\mu \lfloor_{t=0} \le \delta \quad \textit{as well as} \quad \sup_{0 \le t < \min\{T_{f_0, \mathbb{0}}, c\frac{\rho^{-4}}{R^8 + R^4}\}} \rho^{-1} \|f\|_{L^\infty_\mu(\Sigma)} \le R$$

for some $R, \rho > 0$, we have $T_{f_0, \mathbb{0}} > c\frac{\rho^{-4}}{R^8 + R^4}$ and in addition

$$\sup_{0 \le t \le \frac{\rho^{-4}}{R^8 + R^4},\, x \in \mathbb{R}^n} \int_{B_\rho(x)} |A|^2 d\mu \le k\delta.$$

Therefore, if an inverse Willmore flow breaks down in finite time $T < \infty$, then it diverges or a quantum δ of the curvature concentrates in space, i.e.

$$r(\tau) := \sup\{\rho > 0 \mid \sup_{x \in \mathbb{R}^n} \int_{B_\rho(x)} |A|^2 d\mu \lfloor_{t=\tau} \le \delta\} \longrightarrow 0 \quad \text{as} \quad \tau \nearrow T.$$

2 Basics

Consider a smooth variation with normal velocity, i.e.

$$f : \Sigma \times [0, T) \to \mathbb{R}^n \quad \text{and } V = V^\perp = \partial_t f.$$

Let X, Y be tangential vectorfields on Σ independent of t and Φ a normal valued l-form along f. We collect some formulae.

$$(\nabla_X A)(Y, Z) = (\nabla_Y A)(X, Z), \quad \nabla H = -\nabla^* A = -2\nabla^* A^0 \tag{2.1}$$

$$K = \frac{1}{2}(|H|^2 - |A|^2) = \frac{1}{4}|H|^2 - \frac{1}{2}|A^0|^2 \tag{2.2}$$

$$\begin{aligned} R^\perp(X, Y)\Phi &= A(e_i, X)\langle A(e_i, Y), \Phi\rangle - A(e_i, Y)\langle A(e_i, X), \Phi\rangle \\ &= A^0(e_i, X)\langle A^0(e_i, Y), \Phi\rangle - A^0(e_i, Y)\langle A^0(e_i, X), \Phi\rangle \end{aligned} \tag{2.3}$$

$$\partial_t P = -\partial_t P^\perp = \langle \nabla_{e_i} V, \cdot\rangle e_i + \langle e_i, \cdot\rangle \nabla_{e_i} V \tag{2.4}$$

$$P(\partial_t \Phi) = -\langle \nabla_{e_i} V, \Phi\rangle e_i \tag{2.5}$$

$$\partial_t^\perp \nabla_X \Phi = \nabla_X \partial_t^\perp \Phi + A(X, e_i)\langle \nabla_{e_i} V, \Phi\rangle + \nabla_{e_i} V \langle A(X, e_i), \Phi\rangle \tag{2.6}$$

$$(\partial_t g)(X, Y) = -2\langle A(X, Y), V\rangle \tag{2.7}$$

$$\partial_t(d\mu) = -\langle H, V\rangle d\mu \tag{2.8}$$

$$\begin{aligned} \partial_t(\nabla_X Y) = &-\langle (\nabla_{e_i} A)(X, Y), V\rangle e_i + \langle A(X, Y), \nabla_{e_i} V\rangle e_i \\ &- \langle A(X, e_i), \nabla_Y V\rangle e_i - \langle A(Y, e_i), \nabla_X V\rangle e_i \end{aligned} \tag{2.9}$$

$$\partial_t^\perp A(X, Y) = \nabla^2_{X,Y} V - A(e_i, X)\langle A(Y, e_i), V\rangle \tag{2.10}$$

$$\partial_t^\perp H = \Delta V + Q(A^0)V + \frac{1}{2}H\langle H, V\rangle, \tag{2.11}$$

where by definition

$$A^0 := A - \frac{1}{2}gH \tag{2.12}$$

is the tracefree part of the second fundamental form A,

$$Q(A_0)\Phi := A^0(e_i, e_j)\langle\ A^0(e_i, e_j), \Phi\rangle \tag{2.13}$$

and ∇^* is the adjoint operator to the normal derivative $\nabla = D^\perp$, i.e.

$$\nabla^* \Phi = -(\nabla_{e_i} \Phi)(e_i, \ldots) = -g^{i,j} \nabla_i \Phi_{j,k_2,\ldots,k_l}$$

for any normal valued l-form along f.

Definition 2.1. *Let Φ, Ψ be normal valued multilinear forms along f. Then let $\Phi * \Psi$ denote any normal or real valued multilinear form along f, depending only on Φ and Ψ in a universal, bi-linear way.*

For a normal valued multilinear form Φ along f we denote by $P_r^m(\Phi)$ any term of the type

$$P_r^m(\Phi) = \sum_{i_1+\ldots+i_r=m} \nabla^{i_1}\Phi * \ldots * \nabla^{i_r}\Phi.$$

3 The corresponding flow

Let $f : \Sigma \times [0,T) \longrightarrow \mathbb{R}^n \setminus \{\mathbb{0}\}$ be a Willmore flow.
By (2.8) and (2.11) we derive for the variation of the Willmore functional

$$\delta W(g)(u) = \partial_t W(g+tu)\lfloor_{t=0} = \frac{1}{2}\int_\Sigma \langle \Delta H_g + Q(A_g^0)H_g, u\rangle d\mu_{\Sigma,g}.$$

Since for the inversion $I : \mathbb{R}^n \setminus \{\mathbb{0}\} \longrightarrow \mathbb{R}^n \setminus \{\mathbb{0}\} : x \longrightarrow \frac{x}{\|x\|^2}$

$$\begin{aligned} dI(x)\cdot dI^T(x) =&\big(\frac{1}{\|x\|^2}(\delta_i^j - 2\frac{x_i\delta^{j,n}x_n}{\|x\|^2})\big)\delta_{j,m}\big(\frac{1}{\|x\|^2}(\delta_k^m - 2\frac{x_k\delta^{m,r}x_r}{\|x\|^2})\big)\delta^{k,l} \\ =&\frac{1}{\|x\|^4}\delta_i^l - \frac{2}{\|x\|^6}(\delta_i^r x_k\delta^{k,l}x_r + x_i x_n\delta^{n,l}) + \frac{4}{\|x\|^8}x_i x_n x_r x_k \delta^{n,r}\delta^{k,l} \\ =& \frac{1}{\|x\|^4} id, \end{aligned} \tag{3.1}$$

we obtain for the Jacobian of $I_\sharp f$, where $I_\sharp f(x) := I(f(x))$ for all $x \in \Sigma$,

$$J(I_\sharp f) = \sqrt{\det(d(I_\sharp f)^T d(I_\sharp f))} = \sqrt{\det\frac{df^T df}{\|f\|^4}} = \frac{1}{\|f\|^4} Jf,$$
$$\text{i.e.}\quad d\mu_{I_\sharp f} = \|f\|^{-4} d\mu_f$$

and, since $I^2 = id$, i.e. $id = d(I^2)(x) = dI(I(x))\cdot dI(x)$ for all $x \in \mathbb{R}^n \setminus \{\mathbb{0}\}$,

$$\|x\|^{-4} dI(I(x)) = \|x\|^{-4} dI^{-1}(x) = dI^T(x).$$

By this and the invariance under inversion of the Willmore functional, cf. proposition 13.6 we conclude for arbitrary $u \in C_0^\infty(\Sigma, \mathbb{R}^n)$

$$
\begin{aligned}
\langle \partial_t I_\sharp f, u\rangle_{L^2_{\mu_f}(\Sigma)} =& \langle dI(f)\cdot \partial_t f, u\rangle_{L^2_{\mu_f}(\Sigma)} = \langle \partial_t f, dI^T(f)u\rangle_{L^2_{\mu_f}(\Sigma)} \\
=& \langle -grad_{L^2_{\mu_f}(\Sigma)} W(f), dI^T(f)u\rangle_{L^2_{\mu_f}(\Sigma)} = -\delta W(f)(dI^T(f)u) \\
=& -\delta W(I_\sharp^2 f)(\|f\|^{-4} dI(I_\sharp f)\cdot u) \\
=& -\partial_t W(I_\sharp(I_\sharp f + t\|f\|^{-4}u))\lfloor_{t=0} \\
=& -\partial_t W(I_\sharp f + t\|f\|^{-4}u)\lfloor_{t=0} = -\delta W(I_\sharp f)(\|f\|^{-4}u) \\
=& -\frac{1}{2}\int_\Sigma \langle \Delta H_{I_\sharp f} + Q(A^0_{I_\sharp f})H_{I_\sharp f}, \|f\|^{-4}u\rangle d\mu_{\Sigma, I_\sharp f} \\
=& \langle -\frac{1}{2}\|I_\sharp f\|^8(\Delta H_{I_\sharp f} + Q(A^0_{I_\sharp f})H_{I_\sharp f}), u\rangle_{L^2_{\mu_f}(\Sigma)}.
\end{aligned}
$$

Passing from $I_\sharp f$ to f we have to solve the quasi-linear, parabolic forth-order evolution equation

$$\partial_t f = -\frac{\|f\|^8}{2}(\Delta H + Q(A^0)H), \tag{3.2}$$

to whose solutions we will refer considering the inverse Willmore flow.

4 Evolution of the curvature

For any normal valued l-form Φ we define the curvature

$$R^l(X,Y)\Phi := R^\perp(X,Y)\Phi(X_1,\ldots,X_l) - \sum_{k=1}^{l}\Phi(X_1,\ldots,R(X,Y)X_k,\ldots,X_l).$$

Using (2.3) and the Gauss equation we deduce

$$R^l(X,Y)\Phi = A * A * \Phi. \tag{4.1}$$

Proposition 4.1. *Let Φ be a normal valued l-form along f. Then we have*

$$(\nabla\nabla^* - \nabla^*\nabla)\Phi = A * A * \Phi - \nabla^* T, \tag{4.2}$$

where $T(X_0,\ldots,X_l) = (\nabla_{X_0}\Phi)(X_1,X_2,\ldots,X_l) - (\nabla_{X_1}\Phi)(X_0,X_2,\ldots,X_l)$.

Proof.

$$\begin{aligned}((\nabla\nabla^* - \nabla^*\nabla)\Phi)_{i_1,\dots,i_l} =& - \nabla_{i_1}(g^{i,j}\nabla_i\Phi_{j,i_2,\dots,i_l}) + g^{i,j}\nabla_i\nabla_j\Phi_{i_1,\dots,i_l}\\ =&g^{i,j}\nabla_i\nabla_j\Phi_{i_1,\dots,i_l} - g^{i,j}\nabla_i\nabla_{i_1}\Phi_{j,i_2,\dots,i_l}\\ &+ g^{i,j}\nabla_i\nabla_{i_1}\Phi_{j,i_2,\dots,i_l} - g^{i,j}\nabla_{i_1}\nabla_i\Phi_{j,i_2,\dots,i_l}\\ =&g^{i,j}\nabla_i(\nabla_j\Phi_{i_1,i_2,\dots,i_l} - \nabla_{i_1}\Phi_{j,i_2,\dots,i_l}) + g^{i,j}R^l_{i,i_1}\Phi_{j,i_2,\dots,i_l}\\ =& - (\nabla^*T)_{i_1,\dots,i_l} + (A * A * \Phi)_{i_1,\dots,i_l},\end{aligned}$$

where the last equality follows by (4.1). □

Taking $\Phi = A$ in (4.2), we get by (2.1)

$$\Delta A = \nabla^2 H + A * A * A \quad (\Delta := -\nabla^*\nabla). \tag{4.3}$$

Considering $\nabla\Phi$, where Φ is a normal valued (l-1)-form in (4.2), we get

$$\Delta(\nabla\Phi) - \nabla(\Delta\Phi) = (\nabla\nabla^* - \nabla^*\nabla)\nabla\Phi = A * A * \nabla\Phi + A * \nabla A * \Phi. \tag{4.4}$$

For inductive reasons we need the following

Proposition 4.2. *Let Φ be a normal valued (l-1)-form along a variation $f : \Sigma \times [0, T) \to \mathbb{R}^n$ with normal velocity $V = \partial_t f$ and $\theta : \Sigma \times [0, T) \to \mathbb{R}$. If $\partial_t^\perp\Phi + \theta\Delta^2\Phi = Y$, then $\Psi = \nabla\Phi$ satisfies*

$$\begin{aligned}\partial_t^\perp\Psi + \theta\Delta^2\Psi = \nabla Y - \nabla\theta\Delta^2\Phi + \theta \sum_{i+j+k=3} \nabla^i A * \nabla^j A * \nabla^k\Phi\\ + \nabla A * V * \Phi + A * \nabla V * \Phi.\end{aligned} \tag{4.5}$$

Proof. We use a local frame $e_1, e_2 \in T(0, 1)$ independent of t, for which $\nabla e_1 = \nabla e_2 = 0$ at a given point of $\Sigma \times [0, T)$. There we have for $X_k \in \{e_1, e_2\}$

$$\begin{aligned}(\partial_t^\perp\Psi)(X_1, \dots, X_l) &= \partial_t^\perp((\nabla_{X_1}\Phi)(X_2, \dots, X_l))\\ &= \partial_t^\perp\nabla_{X_1}(\Phi(X_2, \dots, X_l)) - \partial_t^\perp \sum_{k=2}^{l} \Phi(X_2, \dots, \nabla_{X_1}X_k, \dots, X_l).\end{aligned}$$

For the first term we use (2.6) to infer

$$\begin{aligned}\partial_t^\perp\nabla_{X_1}(\Phi(X_2, \dots, X_l)) =&\nabla_{X_1}\partial_t^\perp(\Phi(X_2, \dots, X_l)) + (A * \nabla V * \Phi)(X_1, \dots, X_l)\\ =&(\nabla\partial_t^\perp\Phi + A * \nabla V * \Phi)(X_1, \dots, X_l),\end{aligned}$$

for the second one (2.9)

$$\begin{aligned}
\partial_t^\perp \sum_{k=2}^{l} \Phi(X_2,\dots,\nabla_{X_1}X_k,\dots,X_l) =& \sum_{k=2}^{l} P^\perp \Phi(X_2,\dots,\partial_t\nabla_{X_1}X_k,\dots,X_l)\\
&+\sum_{k=2}^{l}(\partial_t^\perp\Phi)(X_2,\dots,\nabla_{X_1}X_k,\dots,X_l)\\
=&(\nabla A * V * \Phi + A * \nabla V * \Phi)(X_1,\dots,X_l).
\end{aligned}$$

Therefore

$$\theta\Delta^2\Psi + \partial_t^\perp\Psi - \nabla Y = \theta\Delta^2(\nabla\Phi) - \nabla(\theta\Delta^2\Phi) + \nabla A * V * \Phi + A * \nabla V * \Phi.$$

By (4.4) we have

$$\begin{aligned}
&\theta\Delta^2(\nabla\Phi) - \nabla(\theta\Delta^2\Phi)\\
&\quad=\theta\Delta^2(\nabla\Phi) - \theta\nabla(\Delta^2\Phi) - \nabla\theta\Delta^2\Phi\\
&\quad=\theta[\Delta[\Delta(\nabla\Phi) - \nabla(\Delta\Phi)] + \Delta(\nabla(\Delta\Phi)) - \nabla(\Delta(\Delta\Phi))] - \nabla\theta\Delta^2\Phi\\
&\quad=\theta[\Delta(A * A * \nabla\Phi + A * \nabla A * \Phi)\\
&\qquad + A * A * \nabla(\Delta\Phi) + A * \nabla A * (\Delta\Phi)] - \nabla\theta\Delta^2\Phi\\
&\quad=\theta\sum_{i+j+k=3}\nabla^i A * \nabla^j A * \nabla^k\Phi - \nabla\theta\Delta^2\Phi.
\end{aligned}$$

Collecting terms the proposition follows. ☐

To state the main result of this section, we define abbreviatively

$$(i,j,k)\in I(m) :\Longleftrightarrow k\in\{1,3,5\},\quad i\le m+2,\quad i+j+k=m+5. \tag{4.6}$$

Proposition 4.3. *Under the inverse Willmore flow, i.e. solutions of* (3.2), *we have the evolution equations*

$$\partial_t^\perp(\nabla^m A) + \frac{\|f\|^8}{2}\Delta^2(\nabla^m A) = \sum_{(i,j,k)\in I(m),\, j<m+4} \nabla^i\|f\|^8 * P_k^j(A). \tag{4.7}$$

Proof. For $m=0$ we obtain by (2.10) and (3.2)

$$\begin{aligned}
\partial_t^\perp A =& -\nabla^2\Big(\frac{\|f\|^8}{2}(\Delta H + Q(A^0)H)\Big) - A * A * \Big(\frac{\|f\|^8}{2}(\Delta H + Q(A^0)H)\Big)\\
=& -\frac{\|f\|^8}{2}\nabla^2(\Delta H) + \|f\|^8 P_3^2(A) + \nabla\|f\|^8 * [P_1^3(A) + P_3^1(A)]\\
&+\nabla^2\|f\|^8 * [P_1^2(A) + P_3^0(A)] + \|f\|^8[P_3^2(A) + P_5^0(A)].
\end{aligned}$$

Using (4.3) and (4.4) we derive

$$\begin{aligned}\nabla^2(\Delta H) =& \nabla(\nabla(\Delta H)) = \nabla[\Delta(\nabla H) + A * A * \nabla H + A * \nabla A * H] \\ =& \Delta(\nabla^2 H) + A * A * \nabla^2 H + A * \nabla A * \nabla H + P_3^2(A) \\ =& \Delta(\Delta A + A * A * A) + P_3^2(A) = \Delta^2 A + P_3^2(A).\end{aligned}$$

Consequently

$$\begin{aligned}\partial_t^\perp A =& -\frac{\|f\|^8}{2}\Delta^2 A + \|f\|^8[P_3^2(A) + P_5^0(A)] \\ &+ \nabla\|f\|^8 * [P_1^3(A) + P_3^1(A)] \\ &+ \nabla^2\|f\|^8 * [P_1^2(A) + P_3^0(A)],\end{aligned}$$

which is the required form for $m = 0$.
By (3.2) and (4.5) applied to $\Phi := \nabla^m A, \quad \theta := \frac{\|f\|^8}{2}$ we conclude inductively

$$\begin{aligned}&\partial_t^\perp \nabla^{m+1} A + \frac{\|f\|^8}{2}\Delta^2(\nabla^{m+1} A) \\ &= \nabla[\sum_{(i,j,k)\in I(m),\, j<m+4} \nabla^i\|f\|^8 * P_k^j(A)\ ] - \nabla\frac{\|f\|^8}{2} * \Delta^2(\nabla^m A) \\ &\quad + \frac{\|f\|^8}{2}\sum_{i+j+k=3} \nabla^i A * \nabla^j A * \nabla^k(\nabla^m A) \\ &\quad + \nabla A * (\frac{\|f\|^8}{2}(\Delta H + Q(A^0)H)) * \nabla^m A \\ &\quad + A * \nabla(\frac{\|f\|^8}{2}(\Delta H + Q(A^0)H)) * \nabla^m A \\ &= \sum_{(i,j,k)\in I(m),\, j<m+4} \nabla^{i+1}\|f\|^8 * P_k^j(A) \\ &\quad + \sum_{(i,j,k)\in I(m),\, j<m+4} \nabla^i\|f\|^8 * P_k^{j+1}(A) \\ &\quad + \|f\|^8[P_3^{m+3}(A) + P_5^{m+1}(A)] \\ &\quad + \nabla\|f\|^8 * [P_1^{m+4}(A) + P_3^{m+2}(A) + P_5^m(A)] \\ &= \sum_{(i,j,k)\in I(m+1),\, j<m+5} \nabla^i\|f\|^8 * P_k^j(A).\end{aligned}$$

$\square$

5 Energy type inequalities

Proposition 5.1. *Let* $f : \Sigma \times [0,T) \to \mathbb{R}^n$ *with* $V = V^\perp = \partial_t f$, $\theta : \Sigma \times [0,T) \to \mathbb{R}$ *and* Φ *a normal valued* l*-form along* f*, which satisfies*

$$\partial_t^\perp \Phi + \theta \Delta^2 \Phi = Y.$$

Then for any $\eta : \Sigma \times [0,T) \to \mathbb{R}$ *we have*

$$\begin{aligned}
&\frac{d}{dt}\int_\Sigma \frac{1}{2}\eta|\Phi|^2 d\mu + \int_\Sigma \langle \Delta\Phi, \Delta(\theta\eta\Phi)\rangle d\mu - \int_\Sigma \langle Y, \eta\Phi\rangle d\mu \\
&\quad = \int_\Sigma \eta \sum_{k=1}^{l} \langle A(e_{i_k}, e_j), V\rangle \langle \Phi(e_{i_1}, \ldots, e_{i_k}, \ldots, e_{i_l}), \Phi(e_{i_1}, \ldots, e_j, \ldots, e_{i_l})\rangle d\mu \\
&\qquad - \int_\Sigma \frac{1}{2}|\Phi|^2 \langle H, V\rangle \eta d\mu + \int_\Sigma \frac{1}{2}|\Phi|^2 \partial_t \eta d\mu. \qquad (5.1)
\end{aligned}$$

Proof. Recalling (2.8) we have, using summation over repeated indices,

$$\begin{aligned}
&\partial_t \int_\Sigma \frac{1}{2}\eta|\Phi|^2 d\mu = \partial_t \int_\Sigma \frac{1}{2}\eta \langle \Phi(e_{i_1}, \ldots, e_{i_l}), \Phi(e_{i_l}, \ldots, e_{i_l})\rangle d\mu \\
&\quad = \int_\Sigma \frac{1}{2}\partial_t \eta |\Phi|^2 d\mu - \int_\Sigma \frac{1}{2}\eta|\Phi|^2 \langle H, V\rangle d\mu \\
&\qquad + \int_\Sigma \eta \langle \Phi(e_{i_1}, \ldots, e_{i_l}), \partial_t(\Phi(e_{i_1}, \ldots, e_{i_l}))\rangle d\mu \\
&\quad = \int_\Sigma \frac{1}{2}(\partial_t \eta - \eta\langle H, V\rangle)|\Phi|^2 d\mu \\
&\qquad + \int_\Sigma \eta \langle \Phi(e_{i_1}, \ldots, e_{i_l}), (\partial_t^\perp \Phi)(e_{i_1}, \ldots, e_{i_l})\rangle d\mu \\
&\qquad + \int_\Sigma \eta \sum_{k=1}^{l} \langle \Phi(e_{i_1}, \ldots, e_{i_k}, \ldots, e_{i_l}), \Phi(e_{i_1}, \ldots, \partial_t e_{i_k}, \ldots, e_{i_l})\rangle d\mu \\
&\quad = \int_\Sigma \frac{1}{2}(\partial_t \eta - \eta\langle H, V\rangle)|\Phi|^2 d\mu + \int_\Sigma \eta \langle \Phi, -\theta\Delta^2\Phi + Y\rangle d\mu \\
&\qquad + \int_\Sigma \eta \sum_{k=1}^{l} g(\partial_t e_{i_k}, e_j) \langle \Phi(e_{i_1}, \ldots, e_{i_k}, \ldots, e_{i_l}), \Phi(e_{i_1}, \ldots, e_j, \ldots, e_{i_l})\rangle d\mu.
\end{aligned}$$

The claim follows from (2.7) and symmetry in i_k, j. ∎

Proposition 5.2. *Under the assumptions of the previous proposition let*

$$\theta = \varphi^4,\ \eta = \gamma^s \quad \text{with} \quad 0 \le \varphi, \gamma : \Sigma \times [0,T) \to \mathbb{R} \quad \text{and} \quad s \ge 4.$$

Then we have for some $c > 0$

$$\begin{aligned}
&\frac{d}{dt}\int_\Sigma |\Phi|^2\gamma^s d\mu + \int_\Sigma \varphi^4 |\nabla^2\Phi|^2 \gamma^s d\mu - 2\int_\Sigma \langle Y, \Phi\rangle \gamma^s d\mu \\
&\qquad \le \int_\Sigma \langle A * \Phi * \Phi, V\rangle \gamma^s d\mu + \int_\Sigma |\Phi|^2 s\gamma^{s-1}\partial_t\gamma d\mu \\
&\qquad\quad + c\int_\Sigma \varphi^4 (s^4|\nabla\gamma|^4 + s^2\gamma^2|\nabla^2\gamma|^2 + \gamma^4[\ |A|^4 + |\nabla A|^2\ ])\gamma^{s-4}|\Phi|^2 d\mu \\
&\qquad\quad + c\int_\Sigma |\nabla\varphi|^2(|\nabla\varphi|^2\gamma^2 + s^2\varphi^2|\nabla\gamma|^2)\gamma^{s-2}|\Phi|^2 d\mu \\
&\qquad\quad + c\int_\Sigma |\nabla^2\varphi|^2\varphi^2\gamma^s|\Phi|^2 d\mu.
\end{aligned} \tag{5.2}$$

Proof. From the previous proposition, (5.1), we infer

$$\begin{aligned}
\frac{d}{dt}\int_\Sigma |\Phi|^2\gamma^s d\mu + 2\int_\Sigma \langle \Delta\Phi, \Delta(\varphi^4\gamma^s\Phi)\rangle d\mu - 2\int_\Sigma \langle Y, \gamma^s\Phi\rangle d\mu& \\
\le \int_\Sigma \langle A * \Phi * \Phi, V\rangle \gamma^s d\mu + \int_\Sigma |\Phi|^2 s\gamma^{s-1}\partial_t\gamma d\mu&,
\end{aligned}$$

so that it will be sufficient to prove

$$\begin{aligned}
&\int_\Sigma \varphi^4|\nabla^2\Phi|^2\gamma^s d\mu \\
&\quad \le 2\int_\Sigma \langle \Delta\Phi, \Delta(\varphi^4\gamma^s\Phi)\rangle d\mu \\
&\qquad + c\int_\Sigma \varphi^4(s^4|\nabla\gamma|^4 + s^2\gamma^2|\nabla^2\gamma|^2)\gamma^{s-4}|\Phi|^2 d\mu \\
&\qquad + c\int_\Sigma |\nabla\varphi|^2(|\nabla\varphi|^2\gamma^2 + s^2\varphi^2|\nabla\gamma|^2)\gamma^{s-2}|\Phi|^2 d\mu \\
&\qquad + c\int_\Sigma |\nabla^2\varphi|^2\varphi^2\gamma^s|\Phi|^2 d\mu + c\int_\Sigma \varphi^4(\ |A|^4 + |\nabla A|^2\ )\gamma^s|\Phi|^2 d\mu.
\end{aligned} \tag{5.3}$$

Using Young's inequality we get

$$
\begin{aligned}
\int_\Sigma \varphi^4|\nabla^2\Phi|^2\gamma^s d\mu \leq & \int_\Sigma \langle \nabla^2\Phi, \nabla^2(\varphi^4\gamma^s\Phi)\rangle d\mu \\
& + 2\int_\Sigma |\nabla^2\Phi||\nabla(\varphi^4\gamma^s)||\nabla\Phi| d\mu + \int_\Sigma |\nabla^2\Phi||\Phi||\nabla^2(\varphi^4\gamma^s)| d\mu \\
\leq & \int_\Sigma \langle \nabla^2\Phi, \nabla^2(\varphi^4\gamma^s\Phi)\rangle d\mu + c\int_\Sigma |\nabla^2\Phi||\nabla\varphi|\varphi^3\gamma^s|\nabla\Phi| d\mu \\
& + cs\int_\Sigma |\nabla^2\Phi|\varphi^4|\nabla\gamma|\gamma^{s-1}|\nabla\Phi| d\mu + c\int_\Sigma |\nabla^2\Phi||\Phi||\nabla^2\varphi|\varphi^3\gamma^s d\mu \\
& + c\int_\Sigma |\nabla^2\Phi||\Phi||\nabla\varphi|^2\varphi^2\gamma^s d\mu + cs\int_\Sigma |\nabla^2\Phi||\Phi||\nabla\varphi|\varphi^3|\nabla\gamma|\gamma^{s-1} d\mu \\
& + cs^2\int_\Sigma |\nabla^2\Phi||\Phi|\varphi^4|\nabla\gamma|^2\gamma^{s-2} d\mu + cs\int_\Sigma |\nabla^2\Phi||\Phi|\varphi^4|\nabla^2\gamma|\gamma^{s-1} d\mu \\
\leq & \int_\Sigma \langle \nabla^2\Phi, \nabla^2(\varphi^4\gamma^s\Phi)\rangle d\mu \\
& + \varepsilon\int_\Sigma \varphi^4|\nabla^2\Phi|^2\gamma^s d\mu + c(\varepsilon)\int_\Sigma |\nabla\varphi|^2\varphi^2\gamma^s|\nabla\Phi|^2 d\mu \\
& + c(\varepsilon)s^2\int_\Sigma \varphi^4|\nabla\gamma|^2\gamma^{s-2}|\nabla\Phi|^2 d\mu + c(\varepsilon)\int_\Sigma |\nabla^2\varphi|^2\varphi^2\gamma^s|\Phi|^2 d\mu \\
& + c(\varepsilon)\int_\Sigma |\nabla\varphi|^4\gamma^s|\Phi|^2 d\mu + c(\varepsilon)s^2\int_\Sigma |\nabla\varphi|^2\varphi^2|\nabla\gamma|^2\gamma^{s-2}|\Phi|^2 d\mu \\
& + c(\varepsilon)s^4\int_\Sigma \varphi^4|\nabla\gamma|^4\gamma^{s-4}|\Phi|^2 d\mu + c(\varepsilon)s^2\int_\Sigma \varphi^4|\nabla^2\gamma|^2\gamma^{s-2}|\Phi|^2 d\mu.
\end{aligned}
$$

Hence by absorption with $\varepsilon = \frac{1}{3}$

$$
\begin{aligned}
& \int_\Sigma \varphi^4|\nabla^2\Phi|^2\gamma^s d\mu \\
& \quad \leq \frac{3}{2}\int_\Sigma \langle \nabla^2\Phi, \nabla^2(\varphi^4\gamma^s\Phi)\rangle d\mu + c\int_\Sigma \varphi^4(s^4|\nabla\gamma|^4 + s^2\gamma^2|\nabla^2\gamma|^2)\gamma^{s-4}|\Phi|^2 d\mu \\
& \qquad + c\int_\Sigma |\nabla\varphi|^2(|\nabla\varphi|^2\gamma^2 + s^2\varphi^2|\nabla\gamma|^2)\gamma^{s-2}|\Phi|^2 d\mu + c\int_\Sigma |\nabla^2\varphi|^2\varphi^2\gamma^s|\Phi|^2 d\mu \\
& \qquad + c\int_\Sigma \varphi^2(|\nabla\varphi|^2\gamma^2 + s^2\varphi^2|\nabla\gamma|^2)\gamma^{s-2}|\nabla\Phi|^2 d\mu. \qquad (5.4)
\end{aligned}
$$

In the last term we integrate by parts to get

$$
\begin{aligned}
&\int_\Sigma \varphi^2(|\nabla\varphi|^2\gamma^2+s^2\varphi^2|\nabla\gamma|^2)\gamma^{s-2}|\nabla\Phi|^2 d\mu\\
&\quad\le -\int_\Sigma \varphi^2(|\nabla\varphi|^2\gamma^2+s^2\varphi^2|\nabla\gamma|^2)\gamma^{s-2}\langle\Phi,\Delta\Phi\rangle d\mu\\
&\qquad + c\int_\Sigma |\nabla\Phi||\Phi|[\varphi|\nabla\varphi|^3\gamma^s+\varphi^2|\nabla\varphi||\nabla^2\varphi|\gamma^s\\
&\qquad\qquad + s\varphi^2|\nabla\varphi|^2|\nabla\gamma|\gamma^{s-1}+s^2\varphi^3|\nabla\varphi||\nabla\gamma|^2\gamma^{s-2}\\
&\qquad\qquad + s^2\varphi^4|\nabla\gamma||\nabla^2\gamma|\gamma^{s-2}+s^3\varphi^4|\nabla\gamma|^3\gamma^{s-3}]\,d\mu\\
&\quad\le\varepsilon\int_\Sigma \varphi^4|\nabla^2\Phi|^2\gamma^s d\mu + c(\varepsilon)\int_\Sigma(|\nabla\varphi|^2\gamma^2+s^2\varphi^2|\nabla\gamma|^2)^2\gamma^{s-4}|\Phi|^2 d\mu\\
&\qquad + \varepsilon\int_\Sigma \varphi^2(|\nabla\varphi|^2\gamma^2+s^2\varphi^2|\nabla\gamma|^2)\gamma^{s-2}|\nabla\Phi|^2 d\mu\\
&\qquad + c(\varepsilon)\int_\Sigma |\Phi|^2[\,|\nabla\varphi|^4\gamma^s+\varphi^2|\nabla^2\varphi|^2\gamma^s+s^2\varphi^2|\nabla\varphi|^2|\nabla\gamma|^2\gamma^{s-2}\\
&\qquad\qquad + s^2\varphi^4|\nabla^2\gamma|^2\gamma^{s-2}+s^4\varphi^4|\nabla\gamma|^4\gamma^{s-4}\,]\,d\mu,
\end{aligned}
$$

so that we have by absorption

$$
\begin{aligned}
\int_\Sigma \varphi^2(|\nabla\varphi|^2\gamma^2+s^2\varphi^2|\nabla\gamma|^2)\gamma^{s-2}|\nabla\Phi|^2 d\mu
&\le\varepsilon\int_\Sigma \varphi^4|\nabla^2\Phi|^2\gamma^s d\mu\\
&\quad + c(\varepsilon)\int_\Sigma \varphi^4(s^4|\nabla\gamma|^4+s^2\gamma^2|\nabla^2\gamma|^2)\gamma^{s-4}|\Phi|^2 d\mu\\
&\quad + c(\varepsilon)\int_\Sigma |\nabla\varphi|^2(|\nabla\varphi|^2\gamma^2+s^2\varphi^2|\nabla\gamma|^2)\gamma^{s-2}|\Phi|^2 d\mu\\
&\quad + c(\varepsilon)\int_\Sigma |\nabla^2\varphi|^2\varphi^2\gamma^s|\Phi|^2 d\mu. \qquad (5.5)
\end{aligned}
$$

Plugging (5.5) into (5.4) yields by absorption with $\varepsilon=\frac{1}{7}$ yields

$$
\begin{aligned}
\int_\Sigma \varphi^4 |\nabla^2\Phi|^2\gamma^s d\mu \leq & \frac{7}{4}\int_\Sigma \langle \nabla^2\Phi, \nabla^2(\varphi^4\gamma^s\Phi)\rangle d\mu \\
& + c\int_\Sigma \varphi^4(s^4|\nabla\gamma|^4 + s^2\gamma^2|\nabla^2\gamma|^2)\gamma^{s-4}|\Phi|^2 d\mu \\
& + c\int_\Sigma |\nabla\varphi|^2(|\nabla\varphi|^2\gamma^2 + s^2\varphi^2|\nabla\gamma|^2)\gamma^{s-2}|\Phi|^2 d\mu \\
& + c\int_\Sigma |\nabla^2\varphi|^2\varphi^2\gamma^s|\Phi|^2 d\mu. \qquad (5.6)
\end{aligned}
$$

Next we compute using (4.4)

$$
\begin{aligned}
&\int_\Sigma \langle \nabla^2\Phi, \nabla^2(\varphi^4\gamma^s\Phi)\rangle d\mu \\
&\quad = \int_\Sigma \langle \Delta\Phi, \Delta(\varphi^4\gamma^s\Phi)\rangle d\mu + \int_\Sigma \langle A * A * \nabla\Phi + A * \nabla A * \Phi, \nabla(\varphi^4\gamma^s\Phi)\rangle d\mu \\
&\quad \leq \int_\Sigma \langle \Delta\Phi, \Delta(\varphi^4\gamma^s\Phi)\rangle d\mu \\
&\qquad + c\int_\Sigma \varphi^4|A|^2\gamma^s|\nabla\Phi|^2 + cs\int_\Sigma \varphi^4|A|^2|\nabla\gamma|\gamma^{s-1}|\nabla\Phi||\Phi| d\mu \\
&\qquad + c\int_\Sigma \varphi^3|\nabla\varphi||A|^2\gamma^s|\nabla\Phi||\Phi| d\mu + c\int_\Sigma \varphi^4|A||\nabla A|\gamma^s|\nabla\Phi||\Phi| d\mu \\
&\qquad + cs\int_\Sigma \varphi^4|A||\nabla A||\nabla\gamma|\gamma^{s-1}|\Phi|^2 d\mu + c\int_\Sigma \varphi^3|\nabla\varphi||A||\nabla A|\gamma^s|\Phi|^2 d\mu. \qquad (5.7)
\end{aligned}
$$

Treating the second summand by integration by parts

$$
\begin{aligned}
&\int_\Sigma \varphi^4|A|^2\gamma^s|\nabla\Phi|^2 \\
&\qquad \leq -\int_\Sigma \varphi^4|A|^2\gamma^s\langle\Phi, \Delta\Phi\rangle d\mu + c\int_\Sigma \varphi^4|A||\nabla A|\gamma^s|\nabla\Phi||\Phi| d\mu \\
&\qquad\quad + cs\int_\Sigma \varphi^4|A|^2|\nabla\gamma|\gamma^{s-1}|\nabla\Phi||\Phi| d\mu + c\int_\Sigma \varphi^3|\nabla\varphi||A|^2\gamma^s|\nabla\Phi||\Phi| d\mu
\end{aligned}
$$

$$
\begin{aligned}
\leq & \varepsilon \int_\Sigma \varphi^4 |\nabla^2 \Phi|^2 \gamma^s d\mu + \varepsilon \int_\Sigma \varphi^4 |A|^2 \gamma^s |\nabla \Phi|^2 d\mu \\
& + c(\varepsilon) \int_\Sigma \varphi^2 (|\nabla \varphi|^2 \gamma^2 + s^2 \varphi^2 |\nabla \gamma|^2) \gamma^{s-2} |\nabla \Phi|^2 d\mu \\
& + c(\varepsilon) \int_\Sigma \varphi^4 |A|^4 \gamma^s |\Phi|^2 d\mu + c(\varepsilon) \int_\Sigma \varphi^4 |\nabla A|^2 \gamma^s |\Phi|^2 d\mu
\end{aligned}
$$

we obtain by absorption

$$
\begin{aligned}
\int_\Sigma \varphi^4 |A|^2 \gamma^s |\nabla \Phi|^2 \leq & \varepsilon \int_\Sigma \varphi^4 |\nabla^2 \Phi|^2 \gamma^s d\mu \\
& + c(\varepsilon) \int_\Sigma \varphi^2 (|\nabla \varphi|^2 \gamma^2 + s^2 \varphi^2 |\nabla \gamma|^2) \gamma^{s-2} |\nabla \Phi|^2 d\mu \\
& + c(\varepsilon) \int_\Sigma \varphi^4 [\, |A|^4 + |\nabla A|^2 \,] \gamma^s |\Phi|^2 d\mu.
\end{aligned} \tag{5.8}
$$

The remaining terms in (5.7) are estimated to

$$
\begin{aligned}
& s \int_\Sigma \varphi^4 |A|^2 |\nabla \gamma| \gamma^{s-1} |\nabla \Phi| |\Phi| d\mu + \int_\Sigma \varphi^3 |\nabla \varphi| |A|^2 \gamma^s |\nabla \Phi| |\Phi| d\mu \\
& \quad + \int_\Sigma \varphi^4 |A| |\nabla A| \gamma^s |\nabla \Phi| |\Phi| d\mu + s \int_\Sigma \varphi^4 |A| |\nabla A| |\nabla \gamma| \gamma^{s-1} |\Phi|^2 d\mu \\
& \quad + \int_\Sigma \varphi^3 |\nabla \varphi| |A| |\nabla A| \gamma^s |\Phi|^2 d\mu \\
& \leq \int_\Sigma \varphi^4 |A|^2 \gamma^s |\nabla \Phi|^2 \\
& \quad + c \int_\Sigma \varphi^2 (|\nabla \varphi|^2 \gamma^2 + s^2 \varphi^2 |\nabla \gamma|^2) \gamma^{s-2} |\nabla \Phi|^2 d\mu \\
& \quad + c s^4 \int_\Sigma \varphi^4 |\nabla \gamma|^4 \gamma^{s-4} |\Phi|^2 d\mu + c \int_\Sigma |\nabla \varphi|^4 \gamma^s |\Phi|^2 d\mu \\
& \quad + c \int_\Sigma \varphi^4 [\, |A|^4 + |\nabla A|^2 \,] \gamma^s |\Phi|^2 d\mu.
\end{aligned} \tag{5.9}
$$

Now plugging (5.9) into (5.7), then using (5.8) yields

$$\begin{aligned}
\int_\Sigma \langle \nabla^2\Phi, \nabla^2(\varphi^4\gamma^s\Phi)\rangle d\mu \le{}& \varepsilon \int_\Sigma \varphi^4|\nabla^2\Phi|^2\gamma^s d\mu + \int_\Sigma \langle \Delta\Phi, \Delta(\varphi^4\gamma^s\Phi)\rangle d\mu \\
&+ c(\varepsilon)\int_\Sigma \varphi^2(|\nabla\varphi|^2\gamma^2 + s^2\varphi^2|\nabla\gamma|^2)\gamma^{s-2}|\nabla\Phi|^2 d\mu \\
&+ c\int_\Sigma \varphi^4 s^4|\nabla\gamma|^4\gamma^{s-4}|\Phi|^2 d\mu + c\int_\Sigma |\nabla\varphi|^4\gamma^s|\Phi|^2 d\mu \\
&+ c(\varepsilon)\int_\Sigma \varphi^4(\,|A|^4 + |\nabla A|^2\,)\gamma^s|\Phi|^2 d\mu \\
\le{}& \varepsilon \int_\Sigma \varphi^4|\nabla^2\Phi|^2\gamma^s d\mu + \int_\Sigma \langle \Delta\Phi, \Delta(\varphi^4\gamma^s\Phi)\rangle d\mu \\
&+ c(\varepsilon)\int_\Sigma \varphi^4(s^4|\nabla\gamma|^4 + s^2\gamma^2|\nabla^2\gamma|^2)\gamma^{s-4}|\Phi|^2 d\mu \\
&+ c(\varepsilon)\int_\Sigma |\nabla\varphi|^2(|\nabla\varphi|^2\gamma^2 + s^2\varphi^2|\nabla\gamma|^2)\gamma^{s-2}|\Phi|^2 d\mu \\
&+ c(\varepsilon)\int_\Sigma |\nabla^2\varphi|^2\varphi^2\gamma^s|\Phi|^2 d\mu \\
&+ c(\varepsilon)\int_\Sigma \varphi^4(\,|A|^4 + |\nabla A|^2\,)\gamma^s|\Phi|^2 d\mu, \qquad (5.10)
\end{aligned}$$

where (5.5) was used in the last step.
Inserting the above inequality (5.10) into (5.6) verifies by absorption with $\varepsilon = \frac{1}{14}$ equation (5.3), what completes the proof. □

We now apply the data of the inverse Willmore flow to this proposition. Therefore let $\gamma = \widetilde{\gamma} \circ f$, where $0 \le \widetilde{\gamma} \le 1$ and $\|\widetilde{\gamma}\|_{C^2(\mathbb{R}^n)} \le \Lambda$.
This implies $\nabla\gamma = d\widetilde{\gamma}(f)\cdot\nabla f$ and $\nabla^2\gamma = d^2\widetilde{\gamma}(f)\cdot(\nabla f\otimes\nabla f) + d\widetilde{\gamma}(f)\cdot A$, so that we have by this and (3.2) the (in-)equalities

$$|\gamma| \le \Lambda, \quad |\nabla\gamma| \le \Lambda, \quad |\nabla^2\gamma| \le \Lambda(1+|A|) \ \text{ for } \ \|\widetilde{\gamma}\|_{C^2(\mathbb{R}^n)} \le \Lambda,$$

$$V = \partial_t f = -\frac{\|f\|^8}{2}(\Delta H + Q(A^0)H) = \|f\|^8(P_1^2(A) + P_3^0(A)), \qquad (5.11)$$

$$\partial_t\gamma = d\widetilde{\gamma}(f)(-\frac{\|f\|^8}{2}(\Delta H + Q(A^0)H)) = -\frac{\|f\|^8}{2} d\widetilde{\gamma}(f)(\Delta H + P_3^0(A)),$$

to which we will refer as (5.11).

Proposition 5.3. *For $n, m, \Lambda > 0$ and $s \geq 4$ there exists*

$$c = c(n, m, s, \Lambda) > 0$$

such that, if $f : \Sigma \times [0, T) \to \mathbb{R}^n \setminus \{\mathbb{0}\}$ *is an inverse Willmore flow and* $\gamma = \widetilde{\gamma} \circ f$ *as in (5.11), we have*

$$\begin{aligned}\frac{d}{dt} \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu + \int_\Sigma \frac{\|f\|^8}{4} |\nabla^{m+2} A|^2 \gamma^s d\mu \\ \leq \int_\Sigma \sum_{(i,j,k) \in I(m),\, j<m+4} \nabla^i \|f\|^8 * P_k^j(A) * \nabla^m A \gamma^s d\mu \\ + c \int_{[\gamma>0]} [\, \|f\|^8 \gamma^{s-4} + \|f\|^4 \gamma^s \,] \, |\nabla^m A|^2 d\mu. \end{aligned} \tag{5.12}$$

Proof. Consider (5.2) with $\Phi = \nabla^m A$. From (4.7) we infer

$$\varphi = 2^{-\frac{1}{4}} \|f\|^2 \text{ and } Y = \sum_{(i,j,k) \in I(m),\, j<m+4} \nabla^i \|f\|^8 * P_k^j(A).$$

By (5.11) we then have

$$\begin{aligned} &\int_\Sigma 2\langle Y, \Phi \rangle \gamma^s d\mu + \int_\Sigma \langle A * \Phi * \Phi, V \rangle \gamma^s d\mu + \int_\Sigma |\Phi|^2 s \gamma^{s-1} \partial_t \gamma d\mu \\ &+ c \int_\Sigma \varphi^4 (s^4 |\nabla \gamma|^4 + s^2 \gamma^2 |\nabla^2 \gamma|^2 + \gamma^4 [\, |A|^4 + |\nabla A|^2]\,) \gamma^{s-4} |\Phi|^2 d\mu \\ &+ c \int_\Sigma |\nabla \varphi|^2 (|\nabla \varphi|^2 \gamma^2 + s^2 \varphi^2 |\nabla \gamma|^2) \gamma^{s-2} |\Phi|^2 d\mu + c \int_\Sigma |\nabla^2 \varphi|^2 \varphi^2 \gamma^s |\Phi|^2 d\mu \\ \leq &\int_\Sigma \sum_{(i,j,k) \in I(m),\, j<m+4} \nabla^i \|f\|^8 * P_k^j(A) * \Phi \gamma^s d\mu \\ &+ \int_\Sigma \|f\|^8 A * \nabla^m A * (P_1^2(A) + P_3^0(A)) * \Phi \gamma^s d\mu \\ &- \frac{s}{2} \int_\Sigma \|f\|^8 |\Phi|^2 \gamma^{s-1} d\widetilde{\gamma}(f)(\Delta H + P_3^0(A)) d\mu \\ &+ c \int_{[\gamma>0]} \|f\|^8 (s^4 \Lambda^4 + s^2 \Lambda^2 \gamma^2 (1 + |A|)^2 + \gamma^4 (|A|^4 + |\nabla A|^2)) \gamma^{s-4} |\Phi|^2 d\mu \\ &+ c \int_\Sigma |\nabla \|f\|^2|^2 (|\nabla \|f\|^2|^2 \gamma^2 + s^2 \Lambda^2 \|f\|^4) \gamma^{s-2} |\phi|^2 d\mu \\ &+ c \int_\Sigma \|f\|^4 |\nabla^2 \|f\|^2|^2 \gamma^s |\Phi|^2 d\mu \end{aligned}$$

$$\leq \int_\Sigma \sum_{(i,j,k)\in I(m),\, j<m+4} \nabla^i \|f\|^8 * P^j_k(A) * \Phi\gamma^s d\mu$$
$$- \frac{s}{2}\int_\Sigma \|f\|^8|\Phi|^2\gamma^{s-1} d\widetilde{\gamma}(f)(\Delta H)\, d\mu + c\int_{[\gamma>0]} [\, s^4\Lambda^4\|f\|^8\gamma^{s-4} + \|f\|^4\gamma^s\,]|\Phi|^2 d\mu,$$

where we made use of Young's inequality with $p = \frac{4}{3}$, $q = \frac{1}{4}$ to estimate

$$-\frac{s}{2}\int_\Sigma \|f\|^8|\Phi|^2\gamma^{s-1} d\widetilde{\gamma}(f)(P^0_3(A))d\mu \leq cs\Lambda\int_\Sigma \|f\|^8|A|^3\gamma^{s-1}|\Phi|^2 d\mu$$
$$\leq \int_\Sigma \|f\|^8 P^m_5(A) * \Phi\gamma^s d\mu + cs^4\Lambda^4\int_{[\gamma>0]} \|f\|^8\gamma^{s-4}|\Phi|^2 d\mu.$$

For the second term we get by integration by parts and Young's inequality

$$-\frac{s}{2}\int_\Sigma \|f\|^8|\Phi|^2\gamma^{s-1} d\widetilde{\gamma}(f)(\Delta H)d\mu$$
$$\leq cs\Lambda\int_\Sigma |\nabla(\,\|f\|^8|\Phi|^2\gamma^{s-1}\,)||\nabla A|d\mu + cs\Lambda\int_\Sigma \|f\|^8|\Phi|^2\gamma^{s-1}|\nabla A|d\mu$$
$$+ cs\Lambda\int_\Sigma \|f\|^8|\Phi|^2\gamma^{s-1}|A||\nabla A|d\mu$$
$$\leq cs\Lambda\int_\Sigma \|f\|^7\gamma^{s-1}|\nabla A||\Phi|^2 d\mu + cs\Lambda\int_\Sigma \|f\|^8\gamma^{s-1}|\nabla A||\Phi||\nabla\Phi|d\mu$$
$$+ cs^2\Lambda^2\int_\Sigma \|f\|^8\gamma^{s-2}|\nabla A||\Phi|^2 d\mu + cs\Lambda\int_\Sigma \|f\|^8\gamma^{s-1}|\nabla A||\Phi|^2 d\mu$$
$$+ cs\Lambda\int_\Sigma \|f\|^8\gamma^{s-1}|A||\nabla A||\Phi|^2 d\mu$$
$$\leq \int_\Sigma \sum_{(i,j,k)\in I(m),\, j<m+4} \nabla^i\|f\|^8 * P^j_k(A) * \Phi\gamma^s d\mu$$
$$+ cs^4\Lambda^4\int_{[\gamma>0]} \|f\|^8\gamma^{s-4}|\Phi|^2 d\mu$$
$$+ cs^2\Lambda^2\int_\Sigma \|f\|^6\gamma^{s-2}|\Phi|^2 d\mu$$
$$+ cs^2\Lambda^2\int_\Sigma \|f\|^8\gamma^{s-2}|\Phi|^2 d\mu$$
$$+ cs^2\Lambda^2\int_\Sigma \|f\|^8\gamma^{s-2}|\nabla\Phi|^2 d\mu,$$

since for $\varphi := \|f\|^8|\Phi|^2\gamma^{s-1} \geq 0$

$$\begin{aligned}
0 &= \int_\Sigma g^{i,j}\nabla_i(\varphi d\widetilde{\gamma}(f)(\nabla_j H))d\mu \\
&= \int_\Sigma g^{i,j}\nabla_i\varphi \cdot d\widetilde{\gamma}(f)(\nabla_j H)d\mu + \int_\Sigma g^{i,j}\varphi\nabla_i(d\widetilde{\gamma}(f)(\nabla_j H))d\mu \\
&= \int_\Sigma g^{i,j}\nabla_i\varphi \cdot d\widetilde{\gamma}(f)(\nabla_j H)d\mu + \int_\Sigma g^{i,j}\varphi\partial_i(d\widetilde{\gamma}(f)(\nabla_j H))d\mu \\
&\quad - \int_\Sigma g^{i,j}\varphi\Gamma^k_{i,j}d\widetilde{\gamma}(f)(\nabla_k H)d\mu \\
&= \int_\Sigma g^{i,j}\nabla_i\varphi \cdot d\widetilde{\gamma}(f)(\nabla_j H)d\mu + \int_\Sigma g^{i,j}\varphi\, d^2\widetilde{\gamma}(f)(\partial_i f, \nabla_j H)d\mu \\
&\quad - \int_\Sigma g^{i,j}\varphi d\widetilde{\gamma}(f)(\Gamma^k_{i,j}\nabla_k H)d\mu + \int_\Sigma g^{i,j}\varphi\, d\widetilde{\gamma}(f)(\partial_i\nabla_j H)d\mu,
\end{aligned}$$

i.e. with $\nabla_i\nabla_j H = \partial_i\nabla_j H + g^{n,m}\langle\nabla_j H, A_{i,n}\rangle\partial_m f - \Gamma^k_{i,j}\nabla_k H$

$$\begin{aligned}
0 = &\int_\Sigma g^{i,j}\nabla_i\varphi \cdot d\widetilde{\gamma}(f)(\nabla_j H)d\mu + \int_\Sigma g^{i,j}\varphi\, d^2\widetilde{\gamma}(f)(\partial_i f, \nabla_j H)d\mu \\
&+ \int_\Sigma g^{i,j}\varphi\, d\widetilde{\gamma}(f)(\nabla_i\nabla_j H)d\mu - \int_\Sigma g^{i,j}\varphi\, d\widetilde{\gamma}(f)(g^{n,m}\langle\nabla_j H, A_{i,n}\rangle\partial_m f)d\mu,
\end{aligned}$$

and hence

$$\begin{aligned}
-\int_\Sigma \varphi\, d\widetilde{\gamma}(f)(\Delta H)d\mu \leq &c\Lambda\int_\Sigma |\nabla\varphi||\nabla A|d\mu + c\Lambda\int_\Sigma \varphi|\nabla A|d\mu \\
&+ c\Lambda\int_\Sigma \varphi|\nabla A||A|d\mu.
\end{aligned}$$

By proposition 5.2, $0 \leq \gamma \leq 1 \leq \Lambda$ and Young's inequality it follows

$$\begin{aligned}
&\frac{d}{dt}\int_\Sigma |\Phi|^2\gamma^s d\mu + \int_\Sigma \frac{\|f\|^8}{2}|\nabla^2\Phi|^2\gamma^s d\mu \\
&\quad\leq \int_\Sigma \sum_{(i,j,k)\in I(m),\, j<m+4} \nabla^i\|f\|^8 * P^j_k(A) * \Phi\gamma^s d\mu \\
&\qquad + cs^2\Lambda^2\int_\Sigma \|f\|^8\gamma^{s-2}|\nabla\Phi|^2 d\mu + c\int_\Sigma [s^4\Lambda^4\|f\|^8\gamma^{s-4} + \|f\|^4\gamma^s]|\Phi|^2 d\mu.
\end{aligned}$$

By integration by parts one obtains

$$
\begin{aligned}
cs^2\Lambda^2 &\int_\Sigma \|f\|^8 \gamma^{s-2} |\nabla \Phi|^2 d\mu \\
&\leq \frac{1}{4} \int_\Sigma \|f\|^8 |\nabla^2 \Phi|^2 \gamma^s d\mu + c \int_{[\gamma>0]} [\, s^4 \Lambda^4 \|f\|^8 \gamma^{s-4} + s^2 \Lambda^2 \|f\|^6 \gamma^{s-2} \,] |\Phi|^2 d\mu.
\end{aligned}
$$

Plugging in and absorbing we conclude

$$
\begin{aligned}
\frac{d}{dt} \int_\Sigma |\Phi|^2 \gamma^s d\mu + \frac{1}{4} \int_\Sigma \|f\|^8 |\nabla^2 \Phi|^2 \gamma^s d\mu \\
\leq \int_\Sigma \sum_{(i,j,k) \in I(m),\, j<m+4} \nabla^i \|f\|^8 * P_k^j(A) * \Phi \gamma^s d\mu \\
+ c \int_\Sigma [\, s^4 \Lambda^4 \|f\|^8 \gamma^{s-4} + s^2 \Lambda^2 \|f\|^6 \gamma^{s-2} + \|f\|^4 \gamma^s \,] |\Phi|^2 d\mu.
\end{aligned}
$$

The claim follows by Young's inequality. $\square$

6 Estimates on the right hand side

In this section we estimate the second term on the right-hand side of (5.12). This is unfortunately necessary, since $\int_{[\gamma>0]} \|f\|^8 |\nabla^m A|^2 \gamma^{s-4} d\mu$ can not be controlled trivially, when using Gronwall's inequality.

Proposition 6.1. *For $\varepsilon, n, m, \Lambda > 0$ and $s \geq 2m+4$ there exists*

$$c = c(\varepsilon, m, s, \Lambda) > 0$$

such that, if $f : \Sigma \to \mathbb{R}^n$ is a immersion and $\gamma = \widetilde{\gamma} \circ f$ as in (5.11), we have

$$\begin{aligned}\int_{[\gamma>0]} &\|f\|^8 |\nabla^m A|^2 \gamma^{s-4} d\mu \\ &\leq \varepsilon \int_\Sigma \|f\|^8 |\nabla^{m+2} A|^2 \gamma^s d\mu + \varepsilon \sum_{i=0}^m \int_\Sigma \|f\|^4 |\nabla^i A|^2 \gamma^{s-2(m-i)} d\mu \\ &\quad + c \int_{[\gamma>0]} \|f\|^8 |A|^2 \gamma^{s-2m-4} d\mu. \end{aligned} \tag{6.1}$$

Proof. The case $m = 0$ is trivial. Let $m \geq 1$. Applying corollary 13.11 with $p = 2,\ k = 8,\ u = 0,\ v = 1$ we obtain

$$\begin{aligned}\int_\Sigma \|f\|^8 |\nabla \Phi|^2 \gamma^{s-4} d\mu \leq & \varepsilon \int_\Sigma \|f\|^8 |\nabla^2 \Phi|^2 \gamma^{s-2} d\mu \\ & + c(\varepsilon, s, \Lambda) \int_{[\gamma>0]} (\|f\|^6 |\Phi|^2 \gamma^{s-4} + \|f\|^8 |\Phi|^2 \gamma^{s-6}) d\mu \\ \leq & \varepsilon \int_\Sigma \|f\|^8 |\nabla^3 A|^2 \gamma^s d\mu \\ & + \varepsilon \int_\Sigma (\|f\|^6 |\nabla \Phi|^2 \gamma^{s-2} + \int_\Sigma \|f\|^8 |\nabla \Phi|^2 \gamma^{s-4}) d\mu \\ & + c(\varepsilon, s, \Lambda) \int_{[\gamma>0]} (\|f\|^6 |\Phi|^2 \gamma^{s-4} + \|f\|^8 |\Phi|^2 \gamma^{s-6}) d\mu. \end{aligned}$$

By Young's inequality and absorption we derive

$$\begin{aligned}\int_\Sigma \|f\|^8 |\nabla\Phi|^2\gamma^{s-4} d\mu \leq & \varepsilon \int_\Sigma \|f\|^8|\nabla^3\Phi|^2\gamma^s d\mu \\ & + \varepsilon \int_\Sigma [\ \|f\|^4|\nabla\Phi|^2\gamma^s + \|f\|^4|\Phi|^2\gamma^{s-2}\ ] d\mu \\ & + c(\varepsilon, s, \Lambda)\int_{[\gamma>0]} \|f\|^8|\Phi|^2\gamma^{s-6} d\mu.\end{aligned}$$

For $m = 1$ choose $\Phi = A$. For $m \geq 2$ we argue by induction via

$$\begin{aligned}\int_\Sigma \|f\|^8|\nabla^m A|^2\gamma^{s-4} d\mu \leq & \varepsilon \int_\Sigma \|f\|^8|\nabla^{m+2}A|^2\gamma^s d\mu \\ & + \varepsilon \int_\Sigma [\ \|f\|^4|\nabla^m A|^2\gamma^s + \|f\|^4|\nabla^{m-1}A|^2\gamma^{s-2}\ ] d\mu \\ & + c(\varepsilon, s, \Lambda)\int_\Sigma \|f\|^8|\nabla^{m-1}A|^2\gamma^{s-6} d\mu \\ \leq & \varepsilon \int_\Sigma \|f\|^8|\nabla^{m+2}A|^2\gamma^s d\mu + \varepsilon \int_\Sigma \|f\|^8|\nabla^{m+1}A|^2\gamma^{s-2} d\mu \\ & + \varepsilon \sum_{i=0}^m \int_\Sigma \|f\|^4|\nabla^i A|^2\gamma^{s-2(m-1)} d\mu \\ & + c(\varepsilon, s, \Lambda)\int_{[\gamma>0]} \|f\|^8|A|^2\gamma^{s-2m-4} d\mu.\end{aligned}$$

Applying lemma 13.11 we derive

$$\begin{aligned}\int_\Sigma \|f\|^8|\nabla^{m+1}A|^2\gamma^{s-2} d\mu \leq & \int_\Sigma \|f\|^8|\nabla^{m+2}A|^2\gamma^s d\mu \\ & + c(s, \Lambda)\int_\Sigma [\ \|f\|^6\gamma^{s-2} + \|f\|^8\gamma^{s-4}\ ]|\nabla^m A|^2 d\mu \\ \leq & \int_\Sigma \|f\|^8|\nabla^{m+2}A|^2\gamma^s d\mu \\ & + c(s, \Lambda)\int_\Sigma [\ \|f\|^8\gamma^{s-4} + \|f\|^4\gamma^s\ ]|\nabla^m A|^2 d\mu.\end{aligned}$$

The claim follows. $\square$

7 Estimates by smallness assumption, m=0

Proposition 7.1. *For $n, \Lambda > 0$ and $s \geq 4$ there exist*

$$c' = c'(n)\,,\; c = c(n, s, \Lambda) > 0$$

such that, if $f : \Sigma \times [0, T) \to \mathbb{R}^n \setminus \{\mathbb{0}\}$ is an inverse Willmore flow and $\gamma = \widetilde{\gamma} \circ f$ as in (5.11), we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Sigma |A|^2\gamma^s d\mu + \int_\Sigma \frac{\|f\|^8}{8}|\nabla^2 A|^2\gamma^s d\mu \\
&\quad \leq c' \int_\Sigma \|f\|^8|A|^6\gamma^s d\mu + c\int_{[\gamma>0]} [\, \|f\|^8\gamma^{s-4} + \|f\|^4\gamma^s \,]|A|^2 d\mu.
\end{aligned}$$

Proof. Recalling (4.6), (5.11) and observing

$$\begin{aligned}
\int_\Sigma \nabla\|f\|^8 * \nabla^3 A * A\gamma^s d\mu \leq c(n)&\int_\Sigma |\nabla^2\|f\|^8||\nabla^2 A||A|\gamma^s d\mu \\
&+ c(n)\int_\Sigma |\nabla\|f\|^8||\nabla^2 A||\nabla A|\gamma^s d\mu \\
&+ c(n)s\Lambda\int_\Sigma |\nabla\|f\|^8||\nabla^2 A||A|\gamma^{s-1} d\mu
\end{aligned}$$

we obtain

$$\begin{aligned}
\int_\Sigma &\sum_{(i,j,k)\in I(0),\, j<4} \nabla^i\|f\|^8 * P_k^j(A) * A\gamma^s d\mu \\
&= \int_\Sigma \|f\|^8[\nabla A * \nabla A * A * A + \nabla^2 A * A * A * A \\
&\qquad\qquad + A * A * A * A * A * A]\gamma^s \\
&\qquad + \nabla\|f\|^8 * [\nabla^3 A * A + \nabla A * A * A * A]\gamma^s \\
&\qquad + \nabla^2\|f\|^8 * [\nabla^2 A * A + A * A * A * A]\gamma^s \, d\mu \\
&\leq c(n)\int_\Sigma |\nabla^2 A|[\, \|f\|^8\gamma^s|A|^3 + \|f\|^7\gamma^s|\nabla A| \\
&\qquad\qquad + s\Lambda\|f\|^7\gamma^{s-1}|A| + \|f\|^7\gamma^s|A|^2 + \|f\|^6\gamma^s|A| \,] \\
&\qquad + |\nabla A|^2\|f\|^8\gamma^s|A|^2 + |\nabla A|\|f\|^7\gamma^s|A|^3 \\
&\qquad + \|f\|^8\gamma^s|A|^6 + \|f\|^7\gamma^s|A|^5 + \|f\|^6\gamma^s|A|^4 \, d\mu. \qquad (7.1)
\end{aligned}$$

For the first summand in (7.1) Young's inequality yields

$$\begin{aligned}
\int_\Sigma |\nabla^2 A| [\, \|f\|^8 \gamma^s |A|^3 + \|f\|^7 \gamma^s |\nabla A| \\
&+ s\Lambda \|f\|^7 \gamma^{s-1} |A| + \|f\|^7 \gamma^s |A|^2 + \|f\|^6 \gamma^s |A| \,]\, d\mu \\
&\leq \varepsilon \int_\Sigma \|f\|^8 |\nabla^2 A|^2 \gamma^s d\mu \\
&+ c(\varepsilon) \int_\Sigma |\nabla A|^2 \|f\|^6 \gamma^s d\mu \\
&+ c(\varepsilon) \int_\Sigma [\, \|f\|^8 \gamma^s |A|^6 + s^2 \Lambda^2 \|f\|^6 \gamma^{s-2} |A|^2 \\
&\qquad + \|f\|^6 \gamma^s |A|^4 + \|f\|^4 \gamma^s |A|^2 \,]\, d\mu.
\end{aligned}$$

By analogous estimate of the third summand, i.e.

$$\int_\Sigma |\nabla A| \|f\|^7 \gamma^s |A|^3 d\mu \leq c \int_\Sigma |\nabla A|^2 \|f\|^6 \gamma^s d\mu + c \int_\Sigma \|f\|^8 \gamma^s |A|^6 d\mu,$$

and inserting in (7.1) we obtain

$$\begin{aligned}
\int_\Sigma \sum_{(i,j,k) \in I(0),\, j<4} \nabla^i \|f\|^8 * P_k^j(A) * A \gamma^s d\mu \\
&\leq \varepsilon \int_\Sigma \|f\|^8 |\nabla^2 A|^2 \gamma^s d\mu \\
&+ c(n,\varepsilon) \int_\Sigma |\nabla A|^2 [\, \|f\|^6 \gamma^s + \|f\|^8 \gamma^s |A|^2 \,] d\mu \\
&+ c(n,\varepsilon) \int_\Sigma [\, \|f\|^8 \gamma^s |A|^6 + \|f\|^7 \gamma^s |A|^5 + s^2 \Lambda^2 \|f\|^6 \gamma^{s-2} |A|^2 \\
&\qquad + \|f\|^6 \gamma^s |A|^4 + \|f\|^4 \gamma^s |A|^2 \,] d\mu \\
&\leq \varepsilon \int_\Sigma \|f\|^8 |\nabla^2 A|^2 \gamma^s d\mu + c(n,\varepsilon) \int_\Sigma \|f\|^8 \gamma^s |A|^6 d\mu \\
&+ c(n,\varepsilon,s,\Lambda) \int_{[\gamma>0]} [\, \|f\|^8 \gamma^{s-4} + \|f\|^4 \gamma^s \,] |A|^2 d\mu \\
&+ c(n,\varepsilon) \int_\Sigma |\nabla A|^2 [\, \|f\|^6 \gamma^s + \|f\|^8 \gamma^s |A|^2] d\mu. \qquad (7.2)
\end{aligned}$$

Next we come to estimate the fourth summand above.

$$\begin{aligned}
\int_\Sigma |\nabla A|^2[\,\|f\|^6\gamma^s + \|f\|^8\gamma^s|A|^2]d\mu& \\
=& - \int_\Sigma \langle\nabla A, A\rangle * \nabla[\,\|f\|^6\gamma^s + \|f\|^8\gamma^s|A|^2]d\mu \\
& - \int_\Sigma \langle\Delta A, A\rangle[\,\|f\|^6\gamma^s + \|f\|^8\gamma^s|A|^2]d\mu \\
\leq& c\int_\Sigma |\nabla A||A|[\,\|f\|^5\gamma^s + s\Lambda\|f\|^6\gamma^{s-1} \\
&\qquad + \|f\|^7\gamma^s|A|^2 + s\Lambda\|f\|^8\gamma^{s-1}|A|^2]d\mu \\
& - 2\int_\Sigma \|f\|^8\gamma^s\langle\nabla A, A\rangle^2 d\mu \\
& + \int_\Sigma |\nabla^2 A||A|[\,\|f\|^6\gamma^s + \|f\|^8\gamma^s|A|^2]d\mu \\
\leq& \varepsilon\int_\Sigma \|f\|^8|\nabla^2 A|^2\gamma^s d\mu + \varepsilon\int_\Sigma |\nabla A|^2[\,\|f\|^6\gamma^s + \|f\|^8\gamma^s|A|^2]d\mu \\
& + c(\varepsilon)\int_\Sigma [\,\|f\|^4\gamma^s|A|^2 + s^2\Lambda^2\|f\|^6\gamma^{s-2}|A|^2 \\
&\qquad + \|f\|^8\gamma^s|A|^6 + s^2\Lambda^2\|f\|^8\gamma^{s-2}|A|^4]d\mu \\
\leq& \varepsilon\int_\Sigma \|f\|^8|\nabla^2 A|^2\gamma^s d\mu + \varepsilon\int_\Sigma |\nabla A|^2[\,\|f\|^6\gamma^s + \|f\|^8\gamma^s|A|^2]d\mu \\
& + c(\varepsilon)\int_\Sigma \|f\|^8|A|^6\gamma^s d\mu \\
& + c(\varepsilon, s, \Lambda)\int_{[\gamma>0]} [\,\|f\|^8\gamma^{s-4} + \|f\|^4\gamma^s]|A|^2 d\mu.
\end{aligned}$$

Absorbing and plugging into (7.2) we conclude

$$\begin{aligned}
\int_\Sigma \sum_{(i,j,k)\in I(0),\, j<4} \nabla^i\|f\|^8 * P^j_k(A) * A\gamma^s d\mu& \\
\leq& \varepsilon\int_\Sigma \|f\|^8|\nabla^2 A|^2\gamma^s d\mu + c(n,\varepsilon)\int_\Sigma \|f\|^8\gamma^s|A|^6 d\mu \\
& + c(n,\varepsilon,s,\Lambda)\int_{[\gamma>0]} [\,\|f\|^8\gamma^{s-4} + \|f\|^4\gamma^s]|A|^2 d\mu.
\end{aligned}$$

Applying this inequality to proposition 5.3, (5.12) proves the claim. ☐

Proposition 7.2. *For $n, \Lambda > 0$ and $s \geq 4$ there exist*

$$\varepsilon_0 = \varepsilon_0(n),\ c_0 = c_0(n, s, \Lambda) > 0$$

such that, if $f : \Sigma \times [0, T) \to \mathbb{R}^n \setminus \{0\}$ *is an inverse Willmore flow,* $\gamma = \widetilde{\gamma} \circ f$ *as in* (5.11) *and*

$$\sup_{0 \leq t < T} \int_{[\gamma > 0]} |A|^2 d\mu \leq \varepsilon_0,$$

we have

$$\int_\Sigma |A|^2 \gamma^s d\mu + \int_0^t \int_\Sigma \frac{\|f\|^8}{16} |\nabla^2 A|^2 \gamma^s d\mu\, dt \leq \int_\Sigma |A|^2 \gamma^s d\mu\lfloor_{t=0}$$
$$+ c_0 \sup_{0 \leq t < T} (\|f\|^8_{L^\infty_\mu([\gamma>0])} + \|f\|^4_{L^\infty_\mu([\gamma>0])}) \cdot t.$$

Proof. We estimate by lemma 13.8

$$\begin{aligned}
\int_\Sigma \|f\|^8 |A|^6 \gamma^s d\mu =& \int_\Sigma (\|f\|^4 |A|^3 \gamma^{\frac{s}{2}})^2 d\mu \\
\leq& c[\int_\Sigma \|f\|^3 |A|^3 \gamma^{\frac{s}{2}} d\mu + \int_\Sigma \|f\|^4 |A|^2 |\nabla A| \gamma^{\frac{s}{2}} d\mu \\
&+ s\Lambda \int_\Sigma \|f\|^4 |A|^3 \gamma^{\frac{s-2}{2}} d\mu + \int_\Sigma \|f\|^4 |A|^4 \gamma^{\frac{s}{2}}\ d\mu\ ]^2 \\
\leq& c \int_{[\gamma>0]} |A|^2 d\mu [\int_\Sigma \|f\|^6 |A|^4 \gamma^s d\mu + \int_\Sigma \|f\|^8 |A|^2 |\nabla A|^2 \gamma^s d\mu \\
&+ s^2 \Lambda^2 \int_\Sigma \|f\|^8 |A|^4 \gamma^{s-2} d\mu + \int_\Sigma \|f\|^8 |A|^6 \gamma^s d\mu].
\end{aligned} \tag{7.3}$$

By integration by parts we get using $\|f\|^8 \langle A, \nabla A \rangle^2 \gamma^s \geq 0$,

$$\begin{aligned}
\int_\Sigma \|f\|^8 |A|^2 |\nabla A|^2 \gamma^s d\mu \leq& c \int_\Sigma \|f\|^7 |A|^3 |\nabla A| \gamma^s d\mu + c \int_\Sigma \|f\|^8 |A|^3 |\nabla^2 A| \gamma^s d\mu \\
&+ cs\Lambda \int_\Sigma \|f\|^8 |A|^3 |\nabla A| \gamma^{s-1} d\mu \\
\leq& \varepsilon \int_\Sigma \|f\|^8 |A|^2 |\nabla A|^2 \gamma^s d\mu + c(\varepsilon) \int_\Sigma \|f\|^6 |A|^4 \gamma^s d\mu \\
&+ c \int_\Sigma \|f\|^8 |\nabla^2 A|^2 \gamma^s d\mu + c \int_\Sigma \|f\|^8 |A|^6 \gamma^s d\mu \\
&+ c(\varepsilon) s^2 \Lambda^2 \int_\Sigma \|f\|^8 |A|^4 \gamma^{s-2} d\mu,
\end{aligned}$$

i.e. by absorption and Young's inequality

$$\begin{aligned}\int_\Sigma \|f\|^8|A|^2|\nabla A|^2\gamma^s d\mu \leq & c\int_\Sigma \|f\|^8|\nabla^2 A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^8|A|^6\gamma^s d\mu \\ & + c(s,\Lambda)\int_{[\gamma>0]}[\, \|f\|^8\gamma^{s-4} + \|f\|^4\gamma^s\,]|A|^2 d\mu.\end{aligned}$$

For the remaining summands in (7.3) we have

$$\begin{aligned}\int_\Sigma \|f\|^6|A|^4\gamma^s d\mu + s^2\Lambda^2\int_\Sigma \|f\|^8|A|^4\gamma^{s-2}d\mu + \int_\Sigma \|f\|^8|A|^6\gamma^s d\mu \\ \leq c\int_\Sigma \|f\|^8|A|^6\gamma^s d\mu + c(s,\Lambda)\int_{[\gamma>0]}[\, \|f\|^8\gamma^{s-4} + \|f\|^4\gamma^s\,]|A|^2 d\mu.\end{aligned}$$

Applying these inequalities to (7.3) we derive

$$\begin{aligned}\int_\Sigma \|f\|^8|A|^6\gamma^s d\mu \leq & c\int_{[\gamma>0]}|A|^2 d\mu \\ & [\int_\Sigma \|f\|^8|\nabla^2 A|^2\gamma^s d\mu + \int_\Sigma \|f\|^8|A|^6\gamma^s d\mu \\ & + c(s,\Lambda)\int_{[\gamma>0]}[\, \|f\|^8\gamma^{s-4} + \|f\|^4\gamma^s\,]|A|^2 d\mu\,].\end{aligned}$$

Hence

$$\begin{aligned}\int_\Sigma \|f\|^8|A|^6\gamma^s d\mu \leq & \varepsilon\int_\Sigma \|f\|^8|\nabla^2 A|^2\gamma^s d\mu \\ & + c(\varepsilon,s,\Lambda)\int_{[\gamma>0]}[\, \|f\|^8\gamma^{s-4} + \|f\|^4\gamma^s\,]|A|^2 d\mu, \qquad (7.4)\end{aligned}$$

provided, that $\int_{[\gamma>0]}|A|^2 d\mu \leq \delta(\varepsilon)$ is small enough.
Applying this inequality to proposition 7.1 with $\varepsilon = \frac{1}{16c'}$ we conclude

$$\begin{aligned}\frac{d}{dt}\int_\Sigma |A|^2\gamma^s d\mu + \int_\Sigma \frac{\|f\|^8}{16}|\nabla^2 A|^2\gamma^s d\mu \\ \leq c(n,s,\Lambda)\int_{[\gamma>0]}[\, \|f\|^8\gamma^{s-4} + \|f\|^4\gamma^s\,]|A|^2 d\mu.\end{aligned}$$

The proposition follows by integration. $\square$

8 Estimates by smallness assumption, m=1

Proposition 8.1. *For $n, \Lambda, R, d, \tau > 0$ and $s \geq 6$ there exist*

$$\varepsilon_1 = \varepsilon_1(n),\ c_1 = c_1(n, s, \Lambda, R, d, \tau) > 0,$$

such that, if $\quad f : \Sigma \times [0, T) \to \mathbb{R}^n \setminus \{\mathbb{0}\}, \quad 0 < T \leq \tau,$
is an inverse Willmore flow, $\gamma = \widetilde{\gamma} \circ f$ *as in* (5.11) *and*

$$\begin{aligned}
&\sup_{0 \leq t < T} \int_{[\gamma > 0]} |A|^2 d\mu \leq \varepsilon_1, \\
&\sup_{0 \leq t < T} \|f\|_{L^\infty_\mu([\gamma > 0])} \leq R, \\
&\int_0^T \int_{[\gamma > 0]} \|f\|^8 |\nabla^2 A|^2 d\mu \, dt \leq d, \\
&\int_0^T \| \, \|f\|^4 A\|^4_{L^\infty_\mu([\gamma > 0])} dt \leq d,
\end{aligned}$$

we have

$$\sup_{0 \leq t < T} \int_\Sigma |\nabla A|^2 \gamma^s d\mu + \int_0^T \int_\Sigma \|f\|^8 |\nabla^3 A|^2 \gamma^s d\mu \; dt \leq c_1 (1 + \int_\Sigma |\nabla A|^2 \gamma^s d\mu \lfloor_{t=0}).$$

Proof. For abbreviative reasons let $c_1 = c_1(n, s, \Lambda, R)$. We will show

$$\begin{aligned}
\sum_{(i,j,k) \in I(1),\, j < 5} &\int_\Sigma \nabla^i \|f\|^8 * P^j_k(A) * \nabla A \, \gamma^s d\mu \\
&\leq \int_\Sigma \frac{\|f\|^8}{16} |\nabla^3 A|^2 d\mu \\
&\quad + c_1 (1 + \int_{[\gamma > 0]} \|f\|^8 |\nabla^2 A|^2 d\mu + \| \, \|f\|^4 A\|^4_{L^\infty_\mu([\gamma > 0])}) \\
&\qquad (1 + \int_\Sigma |\nabla A|^2 \gamma^s d\mu). \qquad (8.1)
\end{aligned}$$

Applying this inequality and proposition 6.1 with $\varepsilon = \frac{1}{8c}$ to proposition 5.3 we obtain

$$\begin{aligned}
\frac{d}{dt} &\int_\Sigma |\nabla A|^2 \gamma^s d\mu + \int_\Sigma \frac{\|f\|^8}{16} |\nabla^3 A|^2 \gamma^s d\mu \\
&\leq c_1 (1 + \int_{[\gamma > 0]} \|f\|^8 |\nabla^2 A|^2 d\mu + \| \, \|f\|^4 A\|^4_{L^\infty_\mu([\gamma > 0])})(1 + \int_\Sigma |\nabla A|^2 \gamma^s d\mu),
\end{aligned}$$

what proves the proposition using Gronwall's inequality, cf. lemma 13.1.

For the reader's convenience we perform this argument. First we have

$$\frac{d}{dt}(1+\int_\Sigma |\nabla A|^2\gamma^s d\mu)$$
$$\leq c_1(1+\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu+\|\,\|f\|^4 A\|^4_{L^\infty_\mu([\gamma>0])})(1+\int_\Sigma |\nabla A|^2\gamma^s d\mu).$$

Integrating and applying Gronwall's inequality we obtain

$$1+\int_\Sigma |\nabla A|^2\gamma^s d\mu$$
$$\leq (1+\int_\Sigma |\nabla A|^2\gamma^s d\mu\lfloor_{t=0})e^{c_1\int_0^t(1+\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu+\|\,\|f\|^4 A\|^4_{L^\infty_\mu([\gamma>0])})dt}$$
$$\leq (1+\int_\Sigma |\nabla A|^2\gamma^s d\mu\lfloor_{t=0})e^{c_1(\tau+2d)}.$$

It follows

$$\sup_{0\leq t<T}\int_\Sigma |\nabla A|^2\gamma^s d\mu \leq c(n,s,\Lambda,R,d,\tau)(1+\int_\Sigma |\nabla A|^2\gamma^s d\mu\lfloor_{t=0}).$$

Secondly we derive by integration and the already proven estimate

$$\int_0^T\int_\Sigma \frac{\|f\|^8}{16}|\nabla^3 A|^2\gamma^s d\mu\,dt \leq \int_\Sigma |\nabla A|^2\gamma^s d\mu\lfloor_{t=0}$$
$$+\,c_1(1+\sup_{0\leq t<T}\int_\Sigma |\nabla A|^2\gamma^s d\mu)$$
$$\int_0^T(1+\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu+\|\,\|f\|^4 A\|^4_{L^\infty_\mu([\gamma>0])})\,dt$$
$$\leq \int_\Sigma |\nabla A|^2\gamma^s d\mu\lfloor_{t=0}+c(n,s,\Lambda,R)c(n,s,\Lambda,R,d,\tau)$$
$$(1+\int_\Sigma |\nabla A|^2\gamma^s d\mu\lfloor_{t=0})(\tau+2d)$$
$$\leq c(n,s,\Lambda,R,d,\tau)(1+\int_\Sigma |\nabla A|^2\gamma^s d\mu\lfloor_{t=0}).$$

This establishes the required estimate, provided, that (8.1) holds true.

To show (8.1) we give adequate estimates for each term of the sum on the left hand side in (8.1).

- $k = 5, j = 1, i = 0$

For this case we start estimating by lemma 13.8 and Hölder's inequality

$$
\begin{aligned}
&\int_\Sigma \|f\|^8 |A|^4 |\nabla A|^2 \gamma^s d\mu = \int_\Sigma (\|f\|^4 |A|^2 |\nabla A| \gamma^{\frac{s}{2}})^2 d\mu \\
&\quad \leq c[\int_\Sigma \|f\|^3 |A|^2 |\nabla A| \gamma^{\frac{s}{2}} d\mu + \int_\Sigma \|f\|^4 |A| |\nabla A|^2 \gamma^{\frac{s}{2}} d\mu \\
&\qquad\quad + \int_\Sigma \|f\|^4 |A|^2 |\nabla^2 A| \gamma^{\frac{s}{2}} d\mu + s\Lambda \int_\Sigma \|f\|^4 |A|^2 |\nabla A| \gamma^{\frac{s-2}{2}} d\mu \\
&\qquad\quad + \int_\Sigma \|f\|^4 |A|^3 |\nabla A| \gamma^{\frac{s}{2}} d\mu \;]^2 \\
&\quad \leq c \int_{[\gamma > 0]} |A|^2 d\mu \\
&\qquad [\int_\Sigma \|f\|^6 |A|^2 |\nabla A|^2 \gamma^s d\mu + \int_\Sigma \|f\|^8 |\nabla A|^4 \gamma^s d\mu \\
&\qquad + \int_\Sigma \|f\|^8 |A|^2 |\nabla^2 A|^2 \gamma^s d\mu + s^2 \Lambda^2 \int_\Sigma \|f\|^8 |A|^2 |\nabla A|^2 \gamma^{s-2} d\mu \\
&\qquad + \int_\Sigma \|f\|^8 |A|^4 |\nabla A|^2 \gamma^s d\mu \;] \\
&\quad \leq c \int_{[\gamma > 0]} |A|^2 d\mu \\
&\qquad [\int_\Sigma \|f\|^8 |A|^4 |\nabla A|^2 \gamma^s d\mu + \int_\Sigma \|f\|^4 |\nabla A|^2 \gamma^s d\mu + \int_\Sigma \|f\|^8 |\nabla A|^4 \gamma^s d\mu \\
&\qquad + s^4 \Lambda^4 \int_{[\gamma > 0]} \|f\|^8 |A|^4 d\mu + \int_\Sigma \|f\|^8 |A|^2 |\nabla^2 A|^2 \gamma^s d\mu \;].
\end{aligned}
$$

By integration by parts

$$
\begin{aligned}
&\int_\Sigma \|f\|^8 |\nabla A|^4 \gamma^s d\mu \\
&\qquad \leq c \int_\Sigma \|f\|^7 |A| |\nabla A|^3 \gamma^s d\mu + c \int_\Sigma \|f\|^8 |A| |\nabla A|^2 |\nabla^2 A| \gamma^s d\mu \\
&\qquad\quad + cs\Lambda \int_\Sigma \|f\|^8 |A| |\nabla A|^3 \gamma^{s-1} d\mu
\end{aligned}
$$

$$
\begin{aligned}
\leq & \varepsilon \int_\Sigma \|f\|^8 |\nabla A|^4 \gamma^s d\mu + c(\varepsilon) \int_\Sigma \|f\|^6 |A|^2 |\nabla A|^2 \gamma^s d\mu \\
& + c(\varepsilon) \int_\Sigma \|f\|^8 |A|^2 |\nabla^2 A|^2 \gamma^s d\mu + c(\varepsilon) s^4 \Lambda^4 \int_{[\gamma>0]} \|f\|^8 |A|^4 d\mu,
\end{aligned}
$$

i.e. by absorption and Young's inequality

$$
\begin{aligned}
\int_\Sigma & \|f\|^8 |\nabla A|^4 \gamma^s d\mu \\
& \leq c \int_\Sigma \|f\|^8 |A|^4 |\nabla A|^2 \gamma^s d\mu + c \int_\Sigma \|f\|^4 |\nabla A|^2 \gamma^s d\mu \\
& \quad + c s^4 \Lambda^4 \int_{[\gamma>0]} \|f\|^8 |A|^4 d\mu + c \int_\Sigma \|f\|^8 |A|^2 |\nabla^2 A|^2 \gamma^s d\mu. \qquad (8.2)
\end{aligned}
$$

Inserting this inequality yields

$$
\begin{aligned}
\int_\Sigma & \|f\|^8 |A|^4 |\nabla A|^2 \gamma^s d\mu \\
& \leq c \int_{[\gamma>0]} |A|^2 d\mu \Big[\int_\Sigma \|f\|^8 |A|^4 |\nabla A|^2 \gamma^s d\mu + s^4 \Lambda^4 \int_{[\gamma>0]} \|f\|^8 |A|^4 d\mu \\
& \qquad + \int_\Sigma \|f\|^4 |\nabla A|^2 \gamma^s d\mu + \int_\Sigma \|f\|^8 |A|^2 |\nabla^2 A|^2 \gamma^s d\mu \Big]. \qquad (8.3)
\end{aligned}
$$

Next we have

$$
\begin{aligned}
\int_\Sigma & \|f\|^8 |A|^2 |\nabla^2 A|^2 \gamma^s d\mu = \int_\Sigma (\|f\|^4 |A| |\nabla^2 A| \gamma^{\frac{s}{2}})^2 d\mu \\
& \leq c \Big[\int_\Sigma \|f\|^3 |A| |\nabla^2 A| \gamma^{\frac{s}{2}} d\mu + \int_\Sigma \|f\|^4 |\nabla A| |\nabla^2 A| \gamma^{\frac{s}{2}} d\mu \\
& \qquad + \int_\Sigma \|f\|^4 |A| |\nabla^3 A| \gamma^{\frac{s}{2}} d\mu + s\Lambda \int_\Sigma \|f\|^4 |A| |\nabla^2 A| \gamma^{\frac{s-2}{2}} d\mu \\
& \qquad + \int_\Sigma \|f\|^4 |A|^2 |\nabla^2 A| \gamma^{\frac{s}{2}} d\mu \Big]^2 \\
& \leq c \int_{[\gamma>0]} |A|^2 d\mu \Big[\int_\Sigma \|f\|^8 |\nabla^3 A|^2 \gamma^s d\mu + s^2 \Lambda^2 \int_{[\gamma>0]} \|f\|^8 |\nabla^2 A|^2 d\mu \\
& \qquad + \int_\Sigma \|f\|^6 |\nabla^2 A|^2 \gamma^s d\mu + \int_\Sigma \|f\|^8 |A|^2 |\nabla^2 A|^2 \gamma^s d\mu \Big] \\
& \quad + c \int_{[\gamma>0]} \|f\|^8 |\nabla^2 A|^2 d\mu \int_\Sigma |\nabla A|^2 \gamma^s d\mu. \qquad (8.4)
\end{aligned}
$$

By integration by parts we have

$$\begin{aligned}\int_\Sigma \|f\|^6|\nabla^2 A|^2\gamma^s d\mu \leq & c\int_\Sigma \|f\|^5|\nabla A||\nabla^2 A|\gamma^s d\mu + c\int_\Sigma \|f\|^6|\nabla A||\nabla^3 A|\gamma^s d\mu \\ & + cs\Lambda\int_\Sigma \|f\|^6|\nabla A||\nabla^2 A|\gamma^{s-1} d\mu \\ \leq & \varepsilon\int_\Sigma \|f\|^6|\nabla^2 A|^2\gamma^s d\mu + cs^2\Lambda^2\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu \\ & + c\int_\Sigma \|f\|^8|\nabla^3 A|^2\gamma^s d\mu \\ & + c(\varepsilon)\int_\Sigma \|f\|^4|\nabla A|^2\gamma^s d\mu. \end{aligned} \tag{8.5}$$

Absorbing and inserting (8.5) in (8.4) we obtain, since $\int_{[\gamma>0]}|A|^2 d\mu \leq \varepsilon_1$,

$$\begin{aligned}\int_\Sigma \|f\|^8|A|^2|\nabla^2 A|^2\gamma^s d\mu \leq & c\int_{[\gamma>0]}|A|^2 d\mu\int_\Sigma \|f\|^8|\nabla^3 A|^2\gamma^s d\mu \\ & + c\int_\Sigma \|f\|^4|\nabla A|^2\gamma^s d\mu \\ & + c\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu\int_\Sigma |\nabla A|^2\gamma^s d\mu \\ & + cs^2\Lambda^2\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu. \end{aligned} \tag{8.6}$$

Inserting finally (8.6) in (8.3) we derive

$$\begin{aligned}&\int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu \\ &\leq c\int_{[\gamma>0]}|A|^2 d\mu \\ &\quad [\int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu + \int_\Sigma \|f\|^8|\nabla^3 A|^2\gamma^s d\mu \\ &\quad + \int_\Sigma \|f\|^4|\nabla A|^2\gamma^s d\mu + \int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu\int_\Sigma |\nabla A|^2\gamma^s d\mu \\ &\quad + s^2\Lambda^2\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu + s^4\Lambda^4\int_{[\gamma>0]}\|f\|^8|A|^4 d\mu\,]. \end{aligned}$$

Since $\int_{[\gamma>0]}\|f\|^8|A|^4 d\mu \le c\| \, \|f\|^4 A\|^4_{L^\infty_\mu([\gamma>0])} + c(\int_{[\gamma>0]}|A|^2 d\mu)^2$, we conclude

$$\begin{aligned}
&\int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu \\
&\quad \le c\int_{[\gamma>0]}|A|^2 d\mu[\int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu + \int_\Sigma \|f\|^8|\nabla^3 A|^2\gamma^s d\mu\ ] \\
&\qquad + c_1(1+\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu + \| \, \|f\|^4 A\|^4_{L^\infty_\mu[\gamma>0]})(1+\int_\Sigma |\nabla A|^2\gamma^s d\mu).
\end{aligned}$$

Consequently

$$\begin{aligned}
&\int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu \\
&\qquad \le \varepsilon\int_\Sigma \|f\|^8|\nabla^3 A|^2\gamma^s d\mu \\
&\qquad\quad + c(\varepsilon,c_1)(1+\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu + \| \, \|f\|^4 A\|^4_{L^\infty_\mu([\gamma>0])}) \\
&\qquad\qquad (1+\int_\Sigma |\nabla A|^2\gamma^s d\mu),
\end{aligned} \tag{8.7}$$

provided, that $\int_{[\gamma>0]}|A|^2 d\mu \le \delta(\varepsilon)$.

- $k=5, j=0, i=1$

By Young's inequality we have

$$\int_\Sigma \|f\|^7|A|^5|\nabla A|\gamma^s d\mu \le c\int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^6|A|^6\gamma^s d\mu.$$

From lemma 13.8 we infer

$$\begin{aligned}
\int_\Sigma \|f\|^6|A|^6\gamma^s d\mu =& \int_\Sigma (\|f\|^3|A|^3\gamma^{\frac{s}{2}})^2 d\mu \\
\le& c[\int_\Sigma \|f\|^2|A|^3\gamma^{\frac{s}{2}} d\mu + \int_\Sigma \|f\|^3|A|^2|\nabla A|\gamma^{\frac{s}{2}} d\mu \\
&+ s\Lambda\int_\Sigma \|f\|^3|A|^3\gamma^{\frac{s-2}{2}} d\mu + \int_\Sigma \|f\|^3|A|^4\gamma^{\frac{s}{2}} d\mu]^2 \\
\le& c\int_{[\gamma>0]}|A|^2 d\mu[\int_\Sigma \|f\|^6|A|^6\gamma^s d\mu + \int_{[\gamma>0]}(\|f\|^2 + s^4\Lambda^4\|f\|^6)|A|^2 d\mu \\
&+ \int_\Sigma \|f\|^4|\nabla A|^2\gamma^s d\mu + \int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu].
\end{aligned}$$

Since $\int_{[\gamma>0]}|A|^2 d\mu \leq \varepsilon_1$ we obtain by absorption

$$\int_\Sigma \|f\|^6|A|^6\gamma^s d\mu \leq c\int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu + c_1(1+\int_\Sigma |\nabla A|^2\gamma^s d\mu). \quad (8.8)$$

Reinserting we conclude by (8.7)

$$\begin{aligned}\int_\Sigma \|f\|^7|A|^5|\nabla A|\gamma^s d\mu \leq & c\int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu + c_1(1+\int_\Sigma |\nabla A|^2\gamma^s d\mu)\\ \leq & \varepsilon\int_\Sigma \|f\|^8|\nabla^3 A|^2\gamma^s d\mu\\ & + c(\varepsilon,c_1)(1+\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu + \|\,\|f\|^4 A\|^4_{L^\infty_\mu([\gamma>0])})\\ & (1+\int_\Sigma |\nabla A|^2\gamma^s d\mu),\end{aligned}$$

provided, that $\int_{[\gamma>0]}|A|^2 d\mu \leq \delta(\varepsilon)$.

- $k=3, j=3, i=0$

For the first term we have

$$\begin{aligned}&\int_\Sigma \|f\|^8|A|^2|\nabla A||\nabla^3 A|\gamma^s d\mu\\ &\leq \varepsilon\int_\Sigma \|f\|^8|\nabla^3 A|^2\gamma^s d\mu + c(\varepsilon)\int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu.\end{aligned}$$

Apply (8.7) to obtain a suitable estimate. Next

$$\int_\Sigma \|f\|^8|A||\nabla A|^2|\nabla^2 A|\gamma^s d\mu \leq c\int_\Sigma \|f\|^8|\nabla A|^4\gamma^s d\mu + c\int_\Sigma \|f\|^8|A|^2|\nabla^2 A|^2\gamma^s d\mu.$$

Using (8.2) in a first step, then applying (8.6) we derive

$$\begin{aligned}& c\int_\Sigma \|f\|^8|\nabla A|^4\gamma^s d\mu + c\int_\Sigma \|f\|^8|A|^2|\nabla^2 A|^2\gamma^s d\mu\\ &\leq c\int_{[\gamma>0]}|A|^2 d\mu\int_\Sigma \|f\|^8|\nabla^3 A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu\\ &\quad + c\int_\Sigma \|f\|^4|\nabla A|^2\gamma^s d\mu + c\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu\int_\Sigma |\nabla A|^2\gamma^s d\mu\\ &\quad + cs^4\Lambda^4\int_{[\gamma>0]}\|f\|^8|A|^4 d\mu + cs^2\Lambda^2\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu, \quad (8.9)\end{aligned}$$

hence by (8.7) and $\int_{[\gamma>0]} \|f\|^8|A|^4 d\mu \leq c\| \, \|f\|^4 A\|^4_{L^\infty_\mu([\gamma>0])} + c(\int_{[\gamma>0]} |A|^2 d\mu)^2$

$$\begin{aligned}
\int_\Sigma \|f\|^8|A||\nabla A|^2|\nabla^2 A|\gamma^s d\mu \\
\leq & \varepsilon \int_\Sigma \|f\|^8|\nabla^3 A|^2\gamma^s d\mu \\
& + c(\varepsilon, c_1)(1 + \int_{[\gamma>0]} \|f\|^8|\nabla^2 A|^2 d\mu + \| \, \|f\|^4 A\|^4_{L^\infty_\mu([\gamma>0])}) \\
& \quad (1 + \int_\Sigma |\nabla A|^2\gamma^s d\mu),
\end{aligned}$$

provided, that $\int_{[\gamma>0]} |A|^2 d\mu \leq \delta(\varepsilon)$.
For the last term of this case, i.e. $\int_\Sigma \|f\|^8|\nabla A|^4\gamma^s d\mu$, apply (8.9) and (8.7).

- $k = 3, j = 2, i = 1$

By Young's inequality we have

$$\begin{aligned}
\int_\Sigma \|f\|^7|A|^2|\nabla A||\nabla^2 A|\gamma^s d\mu + \int_\Sigma \|f\|^7|A||\nabla A|^3\gamma^s d\mu \\
\leq & c\int_\Sigma \|f\|^6|A|^2|\nabla A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^8|\nabla A|^4\gamma^s d\mu \\
& + c\int_\Sigma \|f\|^8|A|^2|\nabla^2 A|^2\gamma^s d\mu \\
\leq & c\int_\Sigma \|f\|^4|\nabla A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu \\
& + c\int_\Sigma \|f\|^8|A|^2|\nabla^2 A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^8|\nabla A|^4\gamma^s d\mu.
\end{aligned}$$

Apply (8.9) and (8.7).

- $k = 3, j = 1, i = 2$

Apply (8.7) to

$$\begin{aligned}
\int_\Sigma (\|f\|^6 + \|f\|^7|A|)|A|^2|\nabla A|^2\gamma^s d\mu \\
\leq & c\int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^4|\nabla A|^2\gamma^s d\mu.
\end{aligned}$$

- $k = 3, j = 0, i = 3$

By Young's inequality we have

$$\begin{aligned}\int_\Sigma (\|f\|^5 + \|f\|^6|A| + \|f\|^7|A|^2 + \|f\|^7|\nabla A|)|A|^3|\nabla A|\gamma^s d\mu \\ \le c\int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^4|\nabla A|^2\gamma^s d\mu \\ + c\int_{[\gamma>0]} \|f\|^2|A|^2 d\mu + c\int_\Sigma \|f\|^6|A|^6\gamma^s d\mu.\end{aligned}$$

Apply (8.8) and (8.7).

- $k = 1, j = 4, i = 1$

We use integration by parts to derive

$$\begin{aligned}&\int_\Sigma \nabla\|f\|^8 * \nabla^4 A * \nabla A\gamma^s d\mu \\ &= -\int_\Sigma \nabla^2\|f\|^8 * \nabla^3 A * \nabla A\gamma^s d\mu - \int_\Sigma \nabla\|f\|^8 * \nabla^3 A * \nabla^2 A\gamma^s d\mu \\ &\quad - s\int_\Sigma \nabla\|f\|^8 * \nabla^3 A * \nabla A * \nabla\gamma\,\gamma^{s-1} d\mu \\ &\le c(n)\int_\Sigma (\|f\|^6 + \|f\|^7|A|)|\nabla A||\nabla^3 A|\gamma^s d\mu \\ &\quad + c(n)\int_\Sigma \|f\|^7|\nabla^2 A||\nabla^3 A|\gamma^s d\mu \\ &\quad + c(n)s\Lambda\int_\Sigma \|f\|^7|\nabla A||\nabla^3 A|\gamma^{s-1} d\mu \\ &\le \varepsilon\int_\Sigma \|f\|^8|\nabla^3 A|^2\gamma^s d\mu + c(n,\varepsilon)\int_\Sigma \|f\|^4|\nabla A|^2\gamma^s d\mu \\ &\quad + c(n,\varepsilon)\int_\Sigma \|f\|^6|A|^2|\nabla A|^2\gamma^s d\mu + c(n,\varepsilon)\int_\Sigma \|f\|^6|\nabla^2 A|^2\gamma^s d\mu \\ &\quad + c(n,\varepsilon)s^2\Lambda^2\int_\Sigma \|f\|^6|\nabla A|^2\gamma^{s-2} d\mu.\end{aligned}$$

By Young's inequality we have

$$\int_\Sigma \|f\|^6|A|^2|\nabla A|^2\gamma^s d\mu \le c\int_\Sigma \|f\|^8|A|^4|\nabla A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^4|\nabla A|^2\gamma^s d\mu.$$

Furthermore we use corollary 13.11 to derive

$$\begin{aligned}
\int_\Sigma \|f\|^6 |\nabla^2 A|^2 \gamma^s d\mu + s^2\Lambda^2 \int_\Sigma \|f\|^6 |\nabla A|^2 \gamma^{s-2} d\mu \\
\leq & \varepsilon \int_\Sigma \|f\|^8 |\nabla^3 A|^2 \gamma^s d\mu + c(\varepsilon) \int_\Sigma \|f\|^4 |\nabla A|^2 \gamma^s d\mu \\
& + c(\varepsilon, s, \Lambda) \int_\Sigma \|f\|^6 |\nabla A|^2 \gamma^{s-2} d\mu \\
\leq & \varepsilon \int_\Sigma \|f\|^8 |\nabla^3 A|^2 \gamma^s d\mu + c(\varepsilon) \int_\Sigma \|f\|^4 |\nabla A|^2 \gamma^s d\mu \\
& + \delta \int_\Sigma \|f\|^6 |\nabla^2 A|^2 \gamma^s d\mu \\
& + c(\varepsilon, \delta, s, \Lambda) \int_{[\gamma>0]} (\|f\|^6 + \|f\|^4) |A|^2 d\mu,
\end{aligned}$$

i.e. by absorption

$$\begin{aligned}
\int_\Sigma \|f\|^6 |\nabla^2 A|^2 \gamma^s d\mu + s^2\Lambda^2 \int_\Sigma \|f\|^6 |\nabla A|^2 \gamma^{s-2} d\mu \\
\leq & \varepsilon \int_\Sigma \|f\|^8 |\nabla^3 A|^2 \gamma^s d\mu + c(\varepsilon) \int_\Sigma \|f\|^4 |\nabla A|^2 \gamma^s d\mu \\
& + c(\varepsilon, s, \Lambda) \int_{[\gamma>0]} (\|f\|^6 + \|f\|^4) |A|^2 d\mu. \qquad (8.10)
\end{aligned}$$

Inserting yields

$$\begin{aligned}
\int_\Sigma \nabla \|f\|^8 * \nabla^4 A * \nabla A \gamma^s d\mu \\
\leq & \varepsilon \int_\Sigma \|f\|^8 |\nabla^3 A|^2 \gamma^s d\mu + c(n, \varepsilon) \int_\Sigma \|f\|^8 |A|^4 |\nabla A|^2 \gamma^s d\mu \\
& + c(n, \varepsilon) \int_\Sigma \|f\|^4 |\nabla A|^2 \gamma^s d\mu \\
& + c(n, \varepsilon, s, \Lambda) \int_{[\gamma>0]} (\|f\|^6 + \|f\|^4) |A|^2 d\mu.
\end{aligned}$$

Apply (8.7).

- $k=1, j=3, i=2$

By Young's inequality we have

$$\begin{aligned}
&\int_\Sigma (\|f\|^6 + \|f\|^7|A|)|\nabla A||\nabla^3 A|\gamma^s d\mu \\
&\qquad \le \varepsilon \int_\Sigma \|f\|^8 |\nabla^3 A|^2 \gamma^s d\mu + c(\varepsilon)\int_\Sigma \|f\|^4 |\nabla A|^2 \gamma^s d\mu \\
&\qquad\quad + c(\varepsilon)\int_\Sigma \|f\|^6 |A|^2 |\nabla A|^2 \gamma^s d\mu \\
&\qquad \le \varepsilon \int_\Sigma \|f\|^8 |\nabla^3 A|^2 \gamma^s d\mu + c(\varepsilon)\int_\Sigma \|f\|^8 |A|^4 |\nabla A|^2 \gamma^s d\mu \\
&\qquad\quad + c(\varepsilon)\int_\Sigma \|f\|^4 |\nabla A|^2 \gamma^s d\mu.
\end{aligned}$$

Apply (8.7).

- $k=1, j=2, i=3$

$$\begin{aligned}
&\int_\Sigma (\|f\|^5 + \|f\|^6|A| + \|f\|^7|A|^2 + \|f\|^7|\nabla A|)|\nabla A||\nabla^2 A|\gamma^s d\mu \\
&\qquad \le c\int_{[\gamma>0]} \|f\|^8|\nabla^2 A|^2 d\mu + c\int_\Sigma (\|f\|^2 + \|f\|^4)|\nabla A|^2\gamma^s d\mu \\
&\qquad\quad + c\int_\Sigma \|f\|^8 |A|^2 |\nabla^2 A|^2 \gamma^s d\mu + c\int_\Sigma \|f\|^8 |A|^4 |\nabla A|^2 \gamma^s d\mu \\
&\qquad\quad + c\int_\Sigma \|f\|^8 |\nabla A|^4 \gamma^s d\mu + c\int_\Sigma \|f\|^6 |\nabla^2 A|^2 \gamma^s d\mu.
\end{aligned}$$

Apply (8.9), (8.7) and (8.10). Collecting terms we conclude

$$\begin{aligned}
&\sum_{(i,j,k)\in I(1),\, j<5} \int_\Sigma \nabla^i \|f\|^8 * P_k^j(A) * \nabla A\, \gamma^s d\mu \\
&\qquad \le \varepsilon \int_\Sigma \|f\|^8 |\nabla^3 A|^2 d\mu \\
&\qquad\quad + c(\varepsilon, c_1)(1 + \int_{[\gamma>0]} \|f\|^8 |\nabla^2 A|^2 d\mu + \| \, \|f\|^4 A\|^4_{L^\infty_\mu([\gamma>0])}) \\
&\qquad\qquad\qquad (1 + \int_\Sigma |\nabla A|^2 \gamma^s d\mu)
\end{aligned}$$

for $\int_{[\gamma>0]} |A|^2 d\mu \le \delta(\varepsilon, n)$ small enough. Choosing $\varepsilon = \frac{1}{16}$ proves (8.1). $\square$

9 Estimates by smallness assumption, m=2

Proposition 9.1. *For $n, \Lambda, R, d, \tau > 0$ and $s \geq 8$ there exist*

$$\varepsilon_2 = \varepsilon_2(n),\ c_2 = c_2(n, s, \Lambda, R, d, \tau) > 0,$$

such that, if $\quad f : \Sigma \times [0, T) \to \mathbb{R}^n \setminus \{\mathbb{0}\}, \quad 0 < T \leq \tau,$
is an inverse Willmore flow, $\gamma = \widetilde{\gamma} \circ f$ *as in* (5.11) *and*

$$\begin{aligned}
&\sup_{0 \leq t < T} \int_{[\gamma > 0]} |A|^2 d\mu \leq \varepsilon_2, \\
&\sup_{0 \leq t < T} \|f\|_{L^\infty_\mu([\gamma > 0])} \leq R, \\
&\sup_{0 \leq t < T} \int_{[\gamma > 0]} |\nabla A|^2 d\mu \leq d, \\
&\int_0^T \int_{[\gamma > 0]} \|f\|^8 |\nabla^k A|^2 d\mu\, dt \leq d \quad \textit{for} \quad k = 2, 3 \quad \textit{and} \\
&\int_0^T \| \, \|f\|^4 \nabla^k A\|^4_{L^\infty_\mu([\gamma > 0])} dt \leq d \quad\ \textit{for } k = 0, 1,
\end{aligned}$$

we have

$$\sup_{0 \leq t < T} \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu + \int_0^T \int_\Sigma \|f\|^8 |\nabla^4 A|^2 \gamma^s d\mu\, dt \leq c_2 (1 + \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu \lfloor_{t=0}).$$

Proof. For abbreviative reasons let $c_2 = c_2(n, s, \Lambda, R, d)$. We will show

$$\begin{aligned}
\sum_{(i,j,k) \in I(2),\, j < 6} &\int_\Sigma \nabla^i \|f\|^8 * P^j_k(A) * \nabla A\, \gamma^s d\mu \\
&\leq \int_\Sigma \frac{\|f\|^8}{16} |\nabla^4 A|^2 d\mu \\
&\quad + c_2 (1 + \int_{[\gamma > 0]} \|f\|^8 |\nabla^2 A|^2 d\mu + \int_{[\gamma > 0]} \|f\|^8 |\nabla^3 A|^2 d\mu \\
&\qquad + \| \, \|f\|^4 A\|^4_{L^\infty_\mu([\gamma > 0])} + \| \, \|f\|^4 \nabla A\|^4_{L^\infty_\mu([\gamma > 0])}) \\
&\qquad (1 + \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu).
\end{aligned} \tag{9.1}$$

Applying this inequality and proposition 6.1 with $\varepsilon = \frac{1}{8c}$ to proposition 5.3 prove the claim using Gronwall's inequality, cf. lemma 13.1.

To show (9.1) we give adequate estimates for each term of the sum.

- $k = 5, j = 2, i = 0$

By lemma 13.8 we estimate

$$
\begin{aligned}
\int_\Sigma \|f\|^8 |A|^4 |\nabla^2 A|^2 \gamma^s d\mu = & \int_\Sigma (\|f\|^4 |A|^2 |\nabla^2 A| \gamma^{\frac{s}{2}})^2 d\mu \\
\leq & c[\int_\Sigma \|f\|^3 |A|^2 |\nabla^2 A| \gamma^{\frac{s}{2}} d\mu + \int_\Sigma \|f\|^4 |A| |\nabla A| |\nabla^2 A| \gamma^{\frac{s}{2}} d\mu \\
& + \int_\Sigma \|f\|^4 |A|^2 |\nabla^3 A| \gamma^{\frac{s}{2}} d\mu + s\Lambda \int_\Sigma \|f\|^4 |A|^2 |\nabla^2 A| \gamma^{\frac{s-2}{2}} d\mu \\
& + \int_\Sigma \|f\|^4 |A|^3 |\nabla^2 A| \gamma^{\frac{s}{2}} d\mu \;]^2 \\
\leq & c \int_{[\gamma>0]} |A|^2 d\mu \\
& [\int_\Sigma \|f\|^6 |A|^2 |\nabla^2 A|^2 \gamma^s d\mu + \int_\Sigma \|f\|^8 |A|^2 |\nabla^3 A|^2 \gamma^s d\mu \\
& + s^2 \Lambda^2 \int_\Sigma \|f\|^8 |A|^2 |\nabla^2 A|^2 \gamma^{s-2} d\mu + \int_\Sigma \|f\|^8 |A|^4 |\nabla^2 A|^2 \gamma^s d\mu \;] \\
& + c \int_{[\gamma>0]} \|f\|^8 |A|^2 |\nabla A|^2 d\mu \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu \\
\leq & c \int_{[\gamma>0]} |A|^2 d\mu \\
& [\int_\Sigma \|f\|^8 |A|^4 |\nabla^2 A|^2 \gamma^s d\mu + \int_\Sigma \|f\|^8 |A|^2 |\nabla^3 A|^2 \gamma^s d\mu \\
& + \int_\Sigma \|f\|^4 |\nabla^2 A|^2 \gamma^s d\mu + s^4 \Lambda^4 \int_{[\gamma>0]} \|f\|^8 |\nabla^2 A|^2 d\mu \;] \\
& + c[\; \| \|f\|^4 A \|^4_{L^\infty_\mu([\gamma>0])} + (\int_{[\gamma>0]} |\nabla A|^2 d\mu)^2 \;] \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu.
\end{aligned}
$$

Since $\int_{[\gamma>0]} |A|^2 d\mu \leq \varepsilon_2$, we obtain by absorption

$$\int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu \leq c\int_{[\gamma>0]}|A|^2 d\mu \int_\Sigma \|f\|^8|A|^2|\nabla^3 A|^2\gamma^s d\mu$$
$$+ c_2(1+\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu + \|\,\|f\|^4 A\|^4_{L^\infty_\mu([\gamma>0])})$$
$$(1+\int_\Sigma |\nabla^2 A|^2\gamma^s d\mu). \tag{9.2}$$

Next we have again by lemma 13.8

$$\int_\Sigma \|f\|^8|A|^2|\nabla^3 A|^2\gamma^s d\mu = \int_\Sigma (\|f\|^4|A||\nabla^3 A|\gamma^{\frac{s}{2}})^2 d\mu$$
$$\leq c\int_{[\gamma>0]}|A|^2 d\mu[\int_\Sigma \|f\|^6|\nabla^3 A|^2\gamma^s d\mu + \int_\Sigma \|f\|^8|\nabla^4 A|^2 d\mu$$
$$+ s^2\Lambda^2\int_{[\gamma>0]}\|f\|^8|\nabla^3 A|^2 d\mu + \int_\Sigma \|f\|^8|A|^2|\nabla^3 A|^2\gamma^s d\mu\,]$$
$$+ c\int_{[\gamma>0]}\|f\|^8|\nabla^3 A|^2 d\mu \int_{[\gamma>0]}|\nabla A|^2 d\mu. \tag{9.3}$$

By corollary 13.11

$$\int_\Sigma \|f\|^6|\nabla^3 A|^2\gamma^s d\mu \leq \varepsilon\int_\Sigma \|f\|^8|\nabla^4 A|^2\gamma^s d\mu + c(\varepsilon)\int_\Sigma \|f\|^4|\nabla^2 A|^2\gamma^s d\mu$$
$$+ c(\varepsilon)s^2\Lambda^2\int_\Sigma \|f\|^6|\nabla^2 A|^2\gamma^{s-2} d\mu$$
$$\leq \varepsilon\int_\Sigma \|f\|^8|\nabla^4 A|^2\gamma^s d\mu + c(\varepsilon)\int_\Sigma \|f\|^4|\nabla^2 A|^2\gamma^s d\mu$$
$$+ c(\varepsilon)s^4\Lambda^4\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu. \tag{9.4}$$

Applying this inequality to (9.3) we derive by absorption

$$\int_\Sigma \|f\|^8|A|^2|\nabla^3 A|^2\gamma^s d\mu \leq c\int_{[\gamma>0]}|A|^2 d\mu \int_\Sigma \|f\|^8|\nabla^4 A|^2 d\mu$$
$$+ c_2(1+\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu + \int_{[\gamma>0]}\|f\|^8|\nabla^3 A|^2 d\mu)(1+\int_\Sigma |\nabla^2 A|^2\gamma^s d\mu). \tag{9.5}$$

Inserting (9.5) in (9.2) we conclude

$$
\begin{aligned}
&\int_\Sigma \|f\|^8 |A|^4 |\nabla^2 A|^2 \gamma^s d\mu \\
&\qquad \le \varepsilon \int_\Sigma \|f\|^8 |\nabla^4 A|^2 d\mu \\
&\qquad\quad + c(\varepsilon, c_2)(1 + \int_{[\gamma>0]} \|f\|^8 |\nabla^2 A|^2 d\mu + \int_{[\gamma>0]} \|f\|^8 |\nabla^3 A|^2 d\mu \\
&\qquad\qquad + \| \|f\|^4 A\|^4_{L^\infty_\mu([\gamma>0])}) \\
&\qquad\qquad (1 + \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu),
\end{aligned}
\tag{9.6}
$$

provided, that $\int_{[\gamma>0]} |A|^2 d\mu \le \delta(\varepsilon)$.
For the next term to be considered we have by Young's inequality

$$
\begin{aligned}
&\int_\Sigma \|f\|^8 |A|^3 |\nabla A|^2 |\nabla^2 A| \gamma^s d\mu \\
&\qquad \le c \int_\Sigma \|f\|^8 |A|^4 |\nabla^2 A|^2 \gamma^s d\mu + c \int_\Sigma \|f\|^8 |A|^2 |\nabla A|^4 \gamma^s d\mu.
\end{aligned}
$$

We estimate by lemma 13.8

$$
\begin{aligned}
\int_\Sigma \|f\|^8 |A|^2 |\nabla A|^4 \gamma^s d\mu =& \int_\Sigma (\|f\|^4 |A| |\nabla A|^2 \gamma^{\frac{s}{2}})^2 d\mu \\
\le& c[\int_\Sigma \|f\|^3 |A| |\nabla A|^2 \gamma^{\frac{s}{2}} d\mu + \int_\Sigma \|f\|^4 |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \\
&+ \int_\Sigma \|f\|^4 |A| |\nabla A| |\nabla^2 A| \gamma^{\frac{s}{2}} d\mu + s\Lambda \int_\Sigma \|f\|^4 |A| |\nabla A|^2 \gamma^{\frac{s-2}{2}} d\mu \\
&+ \int_\Sigma \|f\|^4 |A|^2 |\nabla A|^2 \gamma^{\frac{s}{2}} d\mu\ ]^2 \\
\le& c \int_{[\gamma>0]} |\nabla A|^2 d\mu \\
&[\int_\Sigma \|f\|^6 |A|^2 |\nabla A|^2 \gamma^s d\mu + \int_\Sigma \|f\|^8 |\nabla A|^4 \gamma^s d\mu \\
&+ \int_\Sigma \|f\|^8 |A|^2 |\nabla^2 A|^2 \gamma^s d\mu + s^2 \Lambda^2 \int_\Sigma \|f\|^8 |A|^2 |\nabla A|^2 \gamma^{s-2} d\mu\ ] \\
&+ c \int_{[\gamma>0]} |A|^2 d\mu \int_\Sigma \|f\|^8 |A|^2 |\nabla A|^4 \gamma^s d\mu.
\end{aligned}
$$

Hence by Young's inequality

$$
\begin{aligned}
\int_\Sigma \|f\|^8|A|^2|\nabla A|^4\gamma^s d\mu \leq (\varepsilon + c\int_{[\gamma>0]}|A|^2 d\mu)\int_\Sigma \|f\|^8|A|^2|\nabla A|^4\gamma^s d\mu& \\
+ c\int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu& \\
+ c(\varepsilon, s, \Lambda)(1 + (\int_{[\gamma>0]}|\nabla A|^2 d\mu)^2)& \\
[\int_{[\gamma>0]}\|f\|^4|A|^2 d\mu + \int_{[\gamma>0]}\|f\|^8|\nabla A|^4 d\mu& \\
+ \int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2 d\mu + \int_{[\gamma>0]}\|f\|^8|A|^2 d\mu\ ].&
\end{aligned}
$$

Absorbing, $\int_{[\gamma>0]}\|f\|^8|\nabla A|^4 d\mu \leq c\|\,\|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])} + c(\int_{[\gamma>0]}|\nabla A|^2 d\mu)^2$ and (9.6) yield an adequate estimate.

- $k = 5, j = 1, i = 1$

Clearly

$$
\begin{aligned}
&\int_\Sigma \|f\|^7|A|^4|\nabla A||\nabla^2 A|\gamma^s d\mu \\
&\qquad\leq c\int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^6|A|^4|\nabla A|^2\gamma^s d\mu.
\end{aligned}
$$

By lemma 13.8 we derive

$$
\begin{aligned}
\int_\Sigma \|f\|^6|A|^4|\nabla A|^2\gamma^s d\mu &= \int_\Sigma (\|f\|^3|A|^2|\nabla A|\gamma^{\frac{s}{2}})^2 d\mu \\
\leq c\int_{[\gamma>0]}|A|^2 d\mu& \\
[\int_\Sigma \|f\|^6|A|^4|\nabla A|^2\gamma^s d\mu + \int_{[\gamma>0]}\|f\|^2|\nabla A|^2 d\mu& \\
+ \int_\Sigma \|f\|^6|A|^2|\nabla^2 A|^2\gamma^s d\mu + s^4\Lambda^4\int_{[\gamma>0]}\|f\|^6|\nabla A|^2 d\mu]& \\
+ \varepsilon\int_\Sigma \|f\|^6|A|^4|\nabla A|^2\gamma^s d\mu& \\
+ c(\varepsilon)(\int_{[\gamma>0]}|\nabla A|^2 d\mu)^2\int_\Sigma \|f\|^6|\nabla A|^2\gamma^s d\mu.&
\end{aligned}
$$

Absorbing and reinserting we conclude by Young's inequality

$$\begin{aligned}\int_\Sigma &\|f\|^7|A|^4|\nabla A||\nabla^2 A|\gamma^s d\mu \\ &\le c\int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu + c_2(1+\int_\Sigma |\nabla^2 A|^2\gamma^s d\mu).\end{aligned}$$

Apply (9.6).

- $k=5, j=0, i=2$

First we have

$$\begin{aligned}\int_\Sigma &(\|f\|^6+\|f\|^7|A|)|A|^5|\nabla^2 A|\gamma^s d\mu \\ &\le c\int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^4|A|^6\gamma^s d\mu + c\int_\Sigma \|f\|^6|A|^8\gamma^s d\mu.\end{aligned}$$

By lemma 13.8 we estimate

$$\begin{aligned}\int_\Sigma \|f\|^4|A|^6\gamma^s d\mu &= \int_\Sigma (\|f\|^2|A|^3\gamma^{\frac{s}{2}})^2 d\mu \\ &\le c[\int_\Sigma \|f\||A|^3\gamma^{\frac{s}{2}} d\mu + \int_\Sigma \|f\|^2|A|^2|\nabla A|\gamma^{\frac{s}{2}} d\mu \\ &\qquad + s\Lambda\int_\Sigma \|f\|^2|A|^3\gamma^{\frac{s-2}{2}} d\mu + \int_\Sigma \|f\|^2|A|^4\gamma^{\frac{s}{2}} d\mu\ ]^2 \\ &\le c\int_{[\gamma>0]}|\nabla A|^2 d\mu \int_\Sigma \|f\|^4|A|^4\gamma^s d\mu \\ &\quad + c\int_{[\gamma>0]}|A|^2 d\mu \\ &\qquad [\int_\Sigma \|f\|^2|A|^4\gamma^s d\mu + s^2\Lambda^2\int_\Sigma \|f\|^4|A|^4\gamma^{s-2} d\mu + \int_\Sigma \|f\|^4|A|^6\gamma^s d\mu\ ] \\ &\le (\varepsilon + c\int_{[\gamma>0]}|A|^2 d\mu)\int_\Sigma \|f\|^4|A|^6\gamma^s d\mu + c(\int_{[\gamma>0]}|A|^2 d\mu)^2 \\ &\quad + c(\varepsilon)[\ (\int_{[\gamma>0]}|\nabla A|^2 d\mu)^2 + s^4\Lambda^4\ ]\int_{[\gamma>0]}\|f\|^4|A|^2 d\mu,\end{aligned}$$

i.e. $\int_\Sigma \|f\|^4|A|^6\gamma^s d\mu \le c_2$ by absorption.

Similarly

$$
\begin{aligned}
\int_\Sigma \|f\|^6|A|^8\gamma^s d\mu = & \int_\Sigma (\|f\|^3|A|^4\gamma^{\frac{s}{2}})^2 d\mu \\
\leq & c[\int_\Sigma \|f\|^2|A|^4\gamma^{\frac{s}{2}} d\mu + \int_\Sigma \|f\|^3|A|^3|\nabla A|\gamma^{\frac{s}{2}} d\mu \\
& + s\Lambda \int_\Sigma \|f\|^3|A|^4\gamma^{\frac{s-2}{2}} d\mu + \int_\Sigma \|f\|^3|A|^5\gamma^{\frac{s}{2}} d\mu\ ]^2 \\
\leq & c\int_{[\gamma>0]} |\nabla A|^2 d\mu \int_\Sigma \|f\|^6|A|^6\gamma^s d\mu \\
& + c\int_{[\gamma>0]} |A|^2 d\mu \\
& [\int_\Sigma \|f\|^4|A|^6\gamma^s d\mu + s^2\Lambda^2 \int_\Sigma \|f\|^6|A|^6\gamma^{s-2} d\mu + \int_\Sigma \|f\|^6|A|^8\gamma^s d\mu\ ] \\
\leq & c(\varepsilon + \int_{[\gamma>0]} |A|^2 d\mu) \int_\Sigma \|f\|^6|A|^8\gamma^s d\mu + c(\int_{[\gamma>0]} |A|^2 d\mu)^2 \\
& + c(\varepsilon, s, \Lambda)[\ (\int_{[\gamma>0]} |\nabla A|^2 d\mu)^3 + s^6\Lambda^6\ ] \int_{[\gamma>0]} \|f\|^6|A|^2 d\mu,
\end{aligned}
$$

where Young's inequality with $p = \frac{3}{2}$, $q = 3$ and $6 = \frac{16}{3} + \frac{2}{3}$ was used in the last step. Hence we conclude

$$
\int_\Sigma \|f\|^4|A|^6\gamma^s d\mu + \int_\Sigma \|f\|^6|A|^8\gamma^s d\mu \leq c_2 \tag{9.7}
$$

for $\int_{[\gamma>0]} |A|^2 d\mu \leq \varepsilon_2$ sufficiently small.
Inserting and applying (9.6) yields the required result for this case.

- $k = 3, j = 4, i = 0$

First we have

$$
\begin{aligned}
& \int_\Sigma \|f\|^8|A|^2|\nabla^2 A||\nabla^4 A|\gamma^s d\mu \\
& \qquad \leq \varepsilon \int_\Sigma \|f\|^8|\nabla^4 A|^2\gamma^s d\mu + c(\varepsilon) \int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu.
\end{aligned}
$$

Apply (9.6). Next we estimate

$$\begin{aligned}
&\int_\Sigma \|f\|^8|A||\nabla A||\nabla^2 A||\nabla^3 A|\gamma^s d\mu\\
&\qquad\le c\int_\Sigma \|f\|^8|A|^2|\nabla^3 A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^8|\nabla A|^2|\nabla^2 A|^2\gamma^s d\mu\\
&\qquad\le c\int_\Sigma \|f\|^8|A|^2|\nabla^3 A|^2\gamma^s d\mu + c(1+\|\,\|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])})\int_\Sigma |\nabla^2 A|^2\gamma^s d\mu.
\end{aligned}$$

Apply (9.5). For the third term in this case we have

$$\begin{aligned}
&\int_\Sigma \|f\|^8|A||\nabla^2 A|^3\gamma^s d\mu\\
&\qquad\le \varepsilon\int_\Sigma \|f\|^{12}|\nabla^2 A|^4\gamma^s d\mu + c(\varepsilon)\int_\Sigma \|f\|^4|A|^2|\nabla^2 A|^2\gamma^s d\mu. \qquad (9.8)
\end{aligned}$$

For the first term of the sum above we estimate by integration by parts

$$\begin{aligned}
&\int_\Sigma \|f\|^{12}|\nabla^2 A|^4\gamma^s d\mu\\
&\qquad\le c\int_\Sigma \|f\|^{11}|\nabla A||\nabla^2 A|^3\gamma^s d\mu + c\int_\Sigma \|f\|^{12}|\nabla A||\nabla^2 A|^2|\nabla^3 A|\gamma^s d\mu\\
&\qquad\quad + cs\Lambda\int_\Sigma \|f\|^{12}|\nabla A||\nabla^2 A|^3\gamma^{s-1} d\mu\\
&\qquad\le \varepsilon\int_\Sigma \|f\|^{12}|\nabla^2 A|^4\gamma^s d\mu + c(\varepsilon)\int_{[\gamma>0]} \|f\|^8|\nabla A|^4 d\mu\\
&\qquad\quad + c(\varepsilon)\int_\Sigma \|f\|^{12}|\nabla A|^2|\nabla^3 A|^2\gamma^s d\mu\\
&\qquad\quad + c(\varepsilon)s^4\Lambda^4\int_{[\gamma>0]} \|f\|^{12}|\nabla A|^4 d\mu,
\end{aligned}$$

i.e. by absorption

$$\begin{aligned}
&\int_\Sigma \|f\|^{12}|\nabla^2 A|^4\gamma^s d\mu\\
&\qquad\le c\int_\Sigma \|f\|^{12}|\nabla A|^2|\nabla^3 A|^2\gamma^s d\mu + c_2(1+\|\,\|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])}). \qquad (9.9)
\end{aligned}$$

We proceed using lemma 13.8 to obtain

$$\begin{aligned}
\int_\Sigma \|f\|^{12}|\nabla A|^2|\nabla^3 A|^2\gamma^s d\mu = &\int_\Sigma (\|f\|^6|\nabla A||\nabla^3 A|\gamma^{\frac{s}{2}})^2 d\mu \\
\leq & c[\int_\Sigma \|f\|^5|\nabla A||\nabla^3 A|\gamma^{\frac{s}{2}} d\mu + \int_\Sigma \|f\|^6|\nabla^2 A||\nabla^3 A|\gamma^{\frac{s}{2}} d\mu \\
&+ \int_\Sigma \|f\|^6|\nabla A||\nabla^4 A|\gamma^{\frac{s}{2}} d\mu + s\Lambda\int_\Sigma \|f\|^6|\nabla A||\nabla^3 A|\gamma^{\frac{s-2}{2}} d\mu \\
&+ \int_\Sigma \|f\|^6|A||\nabla A||\nabla^3 A|\gamma^{\frac{s}{2}} d\mu\,]^2 \\
\leq & c\int_{[\gamma>0]} \|f\|^2|\nabla A|^2 d\mu \int_{[\gamma>0]} \|f\|^8|\nabla^3 A|^2 d\mu \\
&+ c\int_{[\gamma>0]} \|f\|^8|\nabla^3 A|^2 d\mu \int_\Sigma \|f\|^4|\nabla^2 A|^2\gamma^s d\mu \\
&+ c\int_{[\gamma>0]} \|f\|^4|\nabla A|^2 d\mu \int_\Sigma \|f\|^8|\nabla^4 A|^2\gamma^s d\mu \\
&+ cs^2\Lambda^2\int_{[\gamma>0]} \|f\|^4|\nabla A|^2 d\mu \int_{[\gamma>0]} \|f\|^8|\nabla^3 A|^2 d\mu \\
&+ c\int_{[\gamma>0]} |A|^2 d\mu \int_\Sigma \|f\|^{12}|\nabla A|^2|\nabla^3 A|^2\gamma^s d\mu,
\end{aligned}$$

i.e. by absorption, since $\int_{[\gamma>0]} |A|^2 d\mu \leq \varepsilon_2$,

$$\begin{aligned}
&\int_\Sigma \|f\|^{12}|\nabla A|^2|\nabla^3 A|^2\gamma^s d\mu \\
&\quad\leq c_2\int_\Sigma \|f\|^8|\nabla^4 A|^2\gamma^s d\mu \\
&\quad\quad + c_2\int_{[\gamma>0]} \|f\|^8|\nabla^3 A|^2 d\mu\,(1+\int_\Sigma |\nabla^2 A|^2\gamma^s d\mu).
\end{aligned} \tag{9.10}$$

Now inserting (9.10) in (9.9) we derive

$$\begin{aligned}
\int_\Sigma \|f\|^{12}|\nabla^2 A|^4\gamma^s d\mu \leq & c_2\int_\Sigma \|f\|^8|\nabla^4 A|^2\gamma^s d\mu \\
&+ c_2(1+\|\,\|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])} + \int_{[\gamma>0]} \|f\|^8|\nabla^3 A|^2 d\mu) \\
&(1+\int_\Sigma |\nabla^2 A|^2\gamma^s d\mu).
\end{aligned} \tag{9.11}$$

Applying (9.11) to (9.8) we finally obtain using Young's inequality

$$\begin{aligned}\int_\Sigma \|f\|^8 |A||\nabla^2 A|^3 \gamma^s d\mu \leq & \varepsilon \int_\Sigma \|f\|^8 |\nabla^4 A|^2 \gamma^s d\mu \\ & + c_2 (1 + \| \|f\|^4 \nabla A\|^4_{L^\infty_\mu([\gamma>0])} + \int_{[\gamma>0]} \|f\|^8 |\nabla^3 A|^2 d\mu) \\ & \quad (1 + \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu) \\ & + c(\varepsilon, c_2) \int_\Sigma \|f\|^8 |A|^4 |\nabla^2 A|^2 \gamma^s d\mu.\end{aligned}$$

Hence (9.6) yields an estimate realizing the full structure of (9.1).
It remains to estimate

$$\int_\Sigma \|f\|^8 |\nabla A|^2 |\nabla^2 A|^2 \gamma^s d\mu \leq c(1 + \| \|f\|^4 \nabla A\|^4_{L^\infty_\mu([\gamma>0])}) \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu.$$

- $k = 3, j = 3, i = 1$

We start estimating

$$\begin{aligned}& \int_\Sigma \|f\|^7 |A|^2 |\nabla^2 A||\nabla^3 A| \gamma^s d\mu \\ & \qquad \leq c \int_\Sigma \|f\|^6 |\nabla^3 A|^2 \gamma^s d\mu + c \int_\Sigma \|f\|^8 |A|^4 |\nabla^2 A|^2 \gamma^s d\mu.\end{aligned}$$

Apply (9.4) and (9.6). Next we have

$$\begin{aligned}& \int_\Sigma \|f\|^7 |A||\nabla A||\nabla^2 A|^2 \gamma^s d\mu \\ & \qquad \leq c(1 + \| \|f\|^4 \nabla A\|^4_{L^\infty_\mu([\gamma>0])}) \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu \\ & \qquad + c \int_\Sigma \|f\|^8 |A|^4 |\nabla^2 A|^2 \gamma^s d\mu + c \int_\Sigma \|f\|^4 |\nabla^2 A|^2 \gamma^s d\mu.\end{aligned}$$

Apply (9.6). Finally

$$\begin{aligned}\int_\Sigma \|f\|^7 |\nabla A|^3 |\nabla^2 A| \gamma^s d\mu \leq & c(1 + \| \|f\|^4 \nabla A\|^4_{L^\infty_\mu([\gamma>0])}) \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu \\ & + c \int_\Sigma \|f\|^6 |\nabla A|^4 \gamma^s d\mu,\end{aligned}$$

where we have by lemma 13.8

$$
\begin{aligned}
\int_\Sigma \|f\|^6 |\nabla A|^4 \gamma^s d\mu = &\int_\Sigma (\|f\|^3 |\nabla A|^2 \gamma^{\frac{s}{2}})^2 d\mu \\
\leq & c[\int_\Sigma \|f\|^2 |\nabla A|^2 \gamma^{\frac{s}{2}} d\mu + \int_\Sigma \|f\|^3 |\nabla A||\nabla^2 A| \gamma^{\frac{s}{2}} d\mu \\
& \quad + s\Lambda \int_\Sigma \|f\|^3 |\nabla A|^2 \gamma^{\frac{s-2}{2}} d\mu + \int_\Sigma \|f\|^3 |A||\nabla A|^2 \gamma^{\frac{s}{2}} d\mu \,]^2 \\
\leq & c_2 (1 + \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu) + c \int_{[\gamma > 0]} |A|^2 d\mu \int_\Sigma \|f\|^6 |\nabla A|^4 \gamma^s d\mu,
\end{aligned}
$$

i.e. by absorption

$$
\int_\Sigma \|f\|^6 |\nabla A|^4 \gamma^s d\mu \leq c_2 (1 + \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu). \tag{9.12}
$$

- $k = 3, j = 2, i = 2$

By Young's inequality we derive

$$
\begin{aligned}
&\int_\Sigma (\|f\|^6 + \|f\|^7 |A|) |A|^2 |\nabla^2 A|^2 \gamma^s d\mu \\
&\qquad \leq c \int_\Sigma \|f\|^8 |A|^4 |\nabla^2 A|^2 \gamma^s d\mu + c \int_\Sigma \|f\|^4 |\nabla^2 A|^2 \gamma^s d\mu.
\end{aligned}
$$

Apply (9.6). For the second and last term of this case likewise

$$
\begin{aligned}
&\int_\Sigma (\|f\|^6 + \|f\|^7 |A|) |A| |\nabla A|^2 |\nabla^2 A| \gamma^s d\mu \\
&\qquad \leq c \int_\Sigma \|f\|^8 |A|^4 |\nabla^2 A|^2 \gamma^s d\mu + c \int_\Sigma \|f\|^6 |\nabla A|^4 \gamma^s d\mu \\
&\qquad \quad + c \int_\Sigma \|f\|^4 |\nabla^2 A|^2 \gamma^s d\mu.
\end{aligned}
$$

Apply (9.6) and (9.12).

- $k=3, j=1, i=3$

Estimating by Young's inequality we have

$$\begin{aligned}
&\int_\Sigma (\|f\|^5+\|f\|^6|A|+\|f\|^7|A|^2+\|f\|^7|\nabla A|)|A|^2|\nabla A||\nabla^2 A|\gamma^s d\mu\\
&\quad\le c\int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu + c\int_{[\gamma>0]}\|f\|^2|\nabla A|^2 d\mu\\
&\qquad + c\int_\Sigma \|f\|^4|A|^2|\nabla A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^6|A|^4|\nabla A|^2\gamma^s d\mu\\
&\qquad + c\int_\Sigma \|f\|^6|\nabla A|^4\gamma^s d\mu\\
&\quad\le c\int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu + c\int_{[\gamma>0]}\|f\|^2|\nabla A|^2 d\mu\\
&\qquad + c\int_{[\gamma>0]}|A|^2 d\mu + c\int_\Sigma \|f\|^4|A|^6\gamma^s d\mu + c\int_\Sigma \|f\|^6|A|^8\gamma^s d\mu\\
&\qquad + c\int_\Sigma \|f\|^6|\nabla A|^4\gamma^s d\mu.
\end{aligned}$$

Apply (9.6), (9.7) and (9.12).

- $k=3, j=0, i=4$

$$\begin{aligned}
&\int_\Sigma (\|f\|^4+\|f\|^5|A|+\|f\|^6|A|^2+\|f\|^6|\nabla A|\\
&\qquad +\|f\|^7|A|^3+\|f\|^7|A||\nabla A|+\|f\|^7|\nabla^2 A|)|A|^3|\nabla^2 A|\gamma^s d\mu\\
&\quad\le c\int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu + c\int_{[\gamma>0]}|A|^2 d\mu + c\int_\Sigma \|f\|^2|A|^4\gamma^s d\mu\\
&\qquad + c\int_\Sigma \|f\|^4|A|^6\gamma^s d\mu + c\int_\Sigma \|f\|^6|A|^8\gamma^s d\mu + c\int_\Sigma \|f\|^4|A|^2|\nabla A|^2\gamma^s d\mu\\
&\qquad + c\int_\Sigma \|f\|^6|A|^4|\nabla A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^6|A|^2|\nabla^2 A|^2\gamma^s d\mu\\
&\quad\le c\int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu + c\int_{[\gamma>0]}|A|^2 d\mu\\
&\qquad + c\int_\Sigma \|f\|^4|A|^6\gamma^s d\mu + c\int_\Sigma \|f\|^6|A|^8\gamma^s d\mu + c\int_\Sigma \|f\|^6|\nabla A|^4\gamma^s d\mu\\
&\qquad + c\int_\Sigma \|f\|^4|\nabla^2 A|^2\gamma^s d\mu.
\end{aligned}$$

Apply (9.6), (9.7) and (9.12).

- $k = 1, j = 5, i = 1$

By integration by parts and Young's inequality we estimate

$$
\begin{aligned}
\int_\Sigma \nabla\|f\|^8 * \nabla^5 A * \nabla^2 A\gamma^s d\mu \\
\le & c(n)\int_\Sigma (\|f\|^6 + \|f\|^7|A|)|\nabla^2 A||\nabla^4 A|\gamma^s d\mu \\
& + c(n)\int_\Sigma \|f\|^7|\nabla^3 A||\nabla^4 A|\gamma^s d\mu \\
& + c(n)s\Lambda \int_\Sigma \|f\|^7|\nabla^2 A||\nabla^4 A|\gamma^{s-1} d\mu \\
\le & \varepsilon \int_\Sigma \|f\|^8|\nabla^4 A|^2\gamma^s d\mu + c(n,\varepsilon)\int_\Sigma \|f\|^4|\nabla^2 A|^2\gamma^s d\mu \\
& + c(n,\varepsilon)\int_\Sigma \|f\|^6|A|^2|\nabla^2 A|^2\gamma^s d\mu + c(n,\varepsilon)\int_\Sigma \|f\|^6|\nabla^3 A|^2\gamma^s d\mu \\
& + c(n,\varepsilon)s^2\Lambda^2 \int_\Sigma \|f\|^6|\nabla^2 A|^2\gamma^{s-2} d\mu \\
\le & \varepsilon \int_\Sigma \|f\|^8|\nabla^4 A|^2\gamma^s d\mu + c(n,\varepsilon)\int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu \\
& + c(n,\varepsilon)\int_\Sigma \|f\|^4|\nabla^2 A|^2\gamma^s d\mu + c(n,\varepsilon,s,\Lambda)\int_{[\gamma>0]} \|f\|^8|\nabla^2 A|^2 d\mu \\
& + c(n,\varepsilon)\int_\Sigma \|f\|^6|\nabla^3 A|^2\gamma^s d\mu.
\end{aligned}
$$

Apply (9.6) and (9.4).

- $k = 1, j = 4, i = 2$

Clearly

$$
\begin{aligned}
\int_\Sigma (\|f\|^6 + \|f\|^7|A|)|\nabla^2 A||\nabla^4 A|\gamma^s d\mu \\
\le & \varepsilon \int_\Sigma \|f\|^8|\nabla^4 A|^2\gamma^s d\mu + c(\varepsilon)\int_\Sigma \|f\|^4|\nabla^2 A|^2\gamma^s d\mu \\
& + c(\varepsilon)\int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu.
\end{aligned}
$$

Apply (9.6).

- $k = 1, j = 3, i = 3$

By Young's inequality we have

$$\begin{aligned}\int_\Sigma (\|f\|^5 + \|f\|^6|A| + \|f\|^7|A|^2 + \|f\|^7|\nabla A|)|\nabla^2 A||\nabla^3 A|\gamma^s d\mu \\ \leq & c\int_\Sigma \|f\|^6|\nabla^3 A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^4|\nabla^2 A|^2\gamma^s d\mu \\ & + c\int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu \\ & + c(1 + \| \|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])})\int_\Sigma |\nabla^2 A|^2\gamma^s d\mu.\end{aligned}$$

Apply (9.4) and (9.6).

- $k = 1, j = 2, i = 4$

Finally

$$\begin{aligned}\int_\Sigma (\|f\|^4 &+ \|f\|^5|A| + \|f\|^6|A|^2 + \|f\|^6|\nabla A| \\ &+ \|f\|^7|A|^3 + \|f\|^7|A||\nabla A| + \|f\|^7|\nabla^2 A|)|\nabla^2 A|^2\gamma^s d\mu \\ \leq & c\int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^4|\nabla^2 A|^2\gamma^s d\mu \\ & + c\| \|f\|^4\nabla A\|_{L^\infty_\mu([\gamma>0])} \int_\Sigma \|f\|^2|\nabla^2 A|^2\gamma^s + \|f\|^3|A||\nabla^2 A|^2\gamma^s d\mu \\ & + \varepsilon\int_\Sigma \|f\|^{12}|\nabla^2 A|^4\gamma^s d\mu + c(\varepsilon)\int_\Sigma \|f\|^2|\nabla^2 A|^2\gamma^s d\mu \\ \leq & c\int_\Sigma \|f\|^8|A|^4|\nabla^2 A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^4|\nabla^2 A|^2\gamma^s d\mu \\ & + c_2(1 + \| \|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])})\int_\Sigma |\nabla^2 A|^2\gamma^s d\mu \\ & + \varepsilon\int_\Sigma \|f\|^{12}|\nabla^2 A|^4\gamma^s d\mu + c(\varepsilon)\int_\Sigma \|f\|^2|\nabla^2 A|^2\gamma^s d\mu.\end{aligned}$$

Apply (9.6) and (9.11). $\square$

10 Further estimates

Proposition 10.1. *For $n, \Lambda, R, d, \tau > 0$, $m \geq 3$ and $s \geq 2m+4$ there exists*

$$c_3 = c_3(n, m, s, \Lambda, R, d, \tau) > 0,$$

such that, if $\quad f : \Sigma \times [0, T) \to \mathbb{R}^n \setminus \{0\}, \quad 0 < T \leq \tau,$
is an inverse Willmore flow, $\gamma = \widetilde{\gamma} \circ f$ as in (5.11) *and*

$$\begin{aligned}
&\sup_{0 \leq t < T} \|f\|_{L^\infty_\mu([\gamma>0])} \leq R, \\
&\sup_{0 \leq t < T} \|A\|_{W^{m-3,\infty}_\mu([\gamma>0])} \leq d, \\
&\int_0^T \| \, \|f\|^4 \nabla^k A\|^4_{L^\infty_\mu([\gamma>0])} dt \leq d \quad \textit{for } k = m-2, m-1, \\
&\sup_{0 \leq t < T} \|A\|_{W^{m-1,2}_\mu([\gamma>0])} \leq d \quad \textit{and} \\
&\int_0^T \int_{[\gamma>0]} \|f\|^8 |\nabla^k A|^2 d\mu \, dt \leq d \quad \textit{for } k = m, m+1,
\end{aligned}$$

we have

$$\begin{aligned}
\sup_{0 \leq t < T} \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu + \int_0^T \int_\Sigma \|f\|^8 |\nabla^{m+2} A|^2 \gamma^s d\mu \, dt& \\
\leq c_3 (1 + \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu \lfloor_{t=0}).&
\end{aligned}$$

Proof. For abbreviative reasons let $c_m = c_m(n, m, s, \Lambda, R, d)$. We will show

$$\begin{aligned}
&\sum_{(i,j,k) \in I(m),\, j < m+4} \int_\Sigma \nabla^i \|f\|^8 * P^j_k(A) * \nabla^m A \, \gamma^s d\mu \\
&\quad \leq \int_\Sigma \frac{\|f\|^8}{16} |\nabla^{m+2} A|^2 \gamma^s d\mu \\
&\qquad + c_m (1 + \int_{[\gamma>0]} \|f\|^8 |\nabla^m A|^2 d\mu + \int_{[\gamma>0]} \|f\|^8 |\nabla^{m+1} A|^2 d\mu \\
&\qquad\qquad + \| \, \|f\|^4 \nabla^{m-2} A\|^4_{L^\infty_\mu([\gamma>0])} + \| \, \|f\|^4 \nabla^{m-1} A\|^4_{L^\infty_\mu([\gamma>0])}) \\
&\qquad\qquad (1 + \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu).
\end{aligned} \tag{10.1}$$

Applying this inequality and proposition 6.1 with $\varepsilon = \frac{1}{8c}$ to proposition 5.3 prove the claim using Gronwall's inequality, cf. lemma 13.1.
To show (10.1) we give adequate estimates for each term of the sum, but due to the $W^{m-3,\infty}$-bounds we only have to examine the cases

$$m-2 \leq j < m+4 \text{ or } m \leq i \leq m+2.$$

- $k=5, j=m, i=0$

We only have to estimate the following four terms, since

$$\sup_{0\leq t<T} \|A\|_{W_\mu^{m-3,\infty}([\gamma>0])} \leq d$$

by assumption. First

$$\begin{aligned}\int_\Sigma \|f\|^8|A|^4|\nabla^m A|^2\gamma^s d\mu \leq& \|f\|^8_{L^\infty_\mu([\gamma>0])}\|A\|^4_{L^\infty_\mu([\gamma>0])}\int_\Sigma |\nabla^m A|^2\gamma^s d\mu\\ \leq& c_m\int_\Sigma |\nabla^m A|^2\gamma^s d\mu.\end{aligned}$$

Next

$$\begin{aligned}&\int_\Sigma \|f\|^8|A|^3|\nabla A||\nabla^{m-1}A||\nabla^m A|\gamma^s d\mu\\ &\quad\leq c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu\\ &\qquad + c\|f\|^8_{L^\infty_\mu([\gamma>0])}\|A\|^6_{L^\infty_\mu([\gamma>0])}(1+\|\,\|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])})\\ &\qquad\qquad\int_{[\gamma>0]}|\nabla^{m-1}A|^2 d\mu\\ &\quad\leq c_m + c_m\|\,\|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])} + c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu\end{aligned}$$

and analogously

$$\begin{aligned}&\int_\Sigma \|f\|^8|A|^3|\nabla^2 A||\nabla^{m-2}A||\nabla^m A|\gamma^s d\mu\\ &\qquad\leq c_m + c_m\|\,\|f\|^4\nabla^2 A\|^4_{L^\infty_\mu([\gamma>0])} + c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu.\end{aligned}$$

Finally

$$\begin{aligned}
\int_\Sigma \|f\|^8 |A|^2 |\nabla A|^2 |\nabla^{m-2} A| |\nabla^m A| \gamma^s d\mu \\
\leq & c \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu \\
& + c \|A\|^4_{L^\infty_\mu([\gamma>0])} \| \, \|f\|^4 \nabla A\|^4_{L^\infty_\mu([\gamma>0])} \int_{[\gamma>0]} |\nabla^{m-2} A|^2 d\mu \\
\leq & c_m \| \, \|f\|^4 \nabla A\|^4_{L^\infty_\mu([\gamma>0])} + c \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu.
\end{aligned}$$

- $k = 5, j = m-1, i = 1$

Clearly we have

$$\begin{aligned}
\int_\Sigma \|f\|^7 |A|^4 |\nabla^{m-1} A| |\nabla^m A| \gamma^s d\mu \\
\leq & c \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu + c \|f\|^{14}_{L^\infty_\mu([\gamma>0])} \|A\|^8_{L^\infty_\mu([\gamma>0])} \int_{[\gamma>0]} |\nabla^{m-1} A|^2 d\mu \\
\leq & c_m + c \int_\Sigma |\nabla^m A| \gamma^s d\mu.
\end{aligned}$$

The second term to be considered is estimated via

$$\begin{aligned}
\int_\Sigma \|f\|^7 |A|^3 |\nabla A| |\nabla^{m-2} A| |\nabla^m A| \gamma^s d\mu \\
\leq & c \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu \\
& + c \|f\|^6_{L^\infty_\mu([\gamma>0])} \|A\|^6_{L^\infty_\mu([\gamma>0])} (1 + \| \, \|f\|^4 \nabla A\|^4_{L^\infty_\mu([\gamma>0])}) \\
& \qquad \int_{[\gamma>0]} |\nabla^{m-2} A|^2 d\mu \\
\leq & c_m + c_m \| \, \|f\|^4 \nabla A\|^4_{L^\infty_\mu([\gamma>0])} + c \int_\Sigma |\nabla^m A| \gamma^s d\mu.
\end{aligned}$$

- $k=5, j=m-2, i=2$

Clearly

$$\begin{aligned}\int_\Sigma (\|f\|^6+\|f\|^7|A|)|A|^4|\nabla^{m-2}A||\nabla^m A|\gamma^s d\mu \\ \leq & c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu \\ & + c(1+\|f\|^{14}_{L^\infty_\mu([\gamma>0])})(1+\|A\|^{10}_{L^\infty_\mu([\gamma>0])})\int_{[\gamma>0]}|\nabla^{m-2}A|^2 d\mu \\ \leq & c_m + c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu.\end{aligned}$$

- $k=5, j=0, i=m$

Note, that it suffices to estimate, cf. corollary 13.4,

$$\begin{aligned}\int_\Sigma \|f\|^7|\nabla^{m-2}A||A|^5|\nabla^m A|\gamma^s d\mu \\ \leq & c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu + c\|f\|^{14}_{L^\infty_\mu([\gamma>0])}\|A\|^{10}_{L^\infty_\mu([\gamma>0])}\int_{[\gamma>0]}|\nabla^{m-2}A|^2 d\mu \\ \leq & c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu + c_m.\end{aligned}$$

- $k=3, j=m+2, i=0$

In this case the following five terms have to be considered.

$$\begin{aligned}\int_\Sigma \|f\|^8|A|^2|\nabla^{m+2}A||\nabla^m A|\gamma^s d\mu \\ \leq & \varepsilon\int_\Sigma \|f\|^8|\nabla^{m+2}A|^2\gamma^s d\mu \\ & + c(\varepsilon)\|f\|^8_{L^\infty_\mu([\gamma>0])}\|A\|^4_{L^\infty_\mu([\gamma>0])}\int_\Sigma |\nabla^m A|^2\gamma^s d\mu.\end{aligned}$$

$$\begin{aligned}\int_\Sigma \|f\|^8|A||\nabla A||\nabla^{m+1}A||\nabla^m A|\gamma^s d\mu \\ \leq & c\int_{[\gamma>0]}\|f\|^8|\nabla^{m+1}A|^2 d\mu \\ & + c\|A\|^2_{L^\infty_\mu([\gamma>0])}(1+\|\,\|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])})\int_\Sigma |\nabla^m A|^2\gamma^s d\mu.\end{aligned}$$

$$\int_\Sigma \|f\|^8(|A||\nabla^2 A| + |\nabla A|^2)|\nabla^m A|^2\gamma^s d\mu$$
$$\le c(1 + \|f\|^4_{L^\infty_\mu([\gamma>0])})(1 + \|A\|_{L^\infty_\mu([\gamma>0])})$$
$$(1 + \| \, \|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])} + \| \, \|f\|^4\nabla^2 A\|^4_{L^\infty_\mu([\gamma>0])}) \int_\Sigma |\nabla^m A|^2\gamma^s d\mu.$$

$$\int_\Sigma \|f\|^8(|A||\nabla^3 A| + |\nabla A||\nabla^2 A|)|\nabla^{m-1}A||\nabla^m A|\gamma^s d\mu$$
$$\le c\|f\|^4_{L^\infty_\mu([\gamma>0])}\|A\|_{L^\infty_\mu([\gamma>0])}(\| \, \|f\|^4\nabla^{m-1}A\|^4_{L^\infty_\mu([\gamma>0])})$$
$$(\int_\Sigma |\nabla^3 A|^2\gamma^s d\mu + \int_\Sigma |\nabla^m A|^2\gamma^s d\mu)$$
$$+ c(1 + \| \, \|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])} + \| \, \|f\|^4\nabla^2 A\|^4_{L^\infty_\mu([\gamma>0])})$$
$$(\int_{[\gamma>0]} |\nabla^{m-1}A|^2 d\mu + \int_\Sigma |\nabla^m A|^2\gamma^s d\mu).$$

$$\int_\Sigma \|f\|^8(|A||\nabla^4 A| + |\nabla A||\nabla^3 A| + |\nabla^2 A|^2)|\nabla^{m-2}A||\nabla^m A|\gamma^s d\mu$$
$$\le c(1 + \| \, \|f\|^4\nabla^{m-2}A\|^4_{L^\infty_\mu([\gamma>0])}) \int_\Sigma |\nabla^m A|^2\gamma^s d\mu$$
$$+ c\|A\|^2_{L^\infty_\mu([\gamma>0])} \int_{[\gamma>0]} \|f\|^8|\nabla^4 A|^2 d\mu$$
$$+ c(1 + \| \, \|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])}) \int_\Sigma |\nabla^3 A|^2\gamma^s d\mu$$
$$+ c(1 + \| \, \|f\|^4\nabla^2 A\|^4_{L^\infty_\mu([\gamma>0])}) \int_{[\gamma>0]} |\nabla^2 A|^2 d\mu.$$

These estimates suffice to establish (10.1), since $m \ge 3$.

- $k = 3, j = m+1, i = 1$

For the first three terms we easily derive

$$\int_\Sigma \|f\|^7|A|^2|\nabla^{m+1}A||\nabla^m A|\gamma^s d\mu$$
$$\le c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu + c\|f\|^6_{L^\infty_\mu([\gamma>0])}\|A\|^4_{L^\infty_\mu([\gamma>0])} \int_{[\gamma>0]} \|f\|^8|\nabla^{m+1}A|^2 d\mu.$$

$$\int_\Sigma \|f\|^7|A||\nabla A||\nabla^m A|^2\gamma^s d\mu$$
$$\leq c\|f\|^3_{L^\infty_\mu([\gamma>0])}\|A\|_{L^\infty_\mu([\gamma>0])}(1+\|\,\|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])})\int_\Sigma |\nabla^m A|^2\gamma^s d\mu.$$

$$\int_\Sigma \|f\|^7|A||\nabla^2 A||\nabla^{m-1}A||\nabla^m A|\gamma^s d\mu$$
$$\leq c(1+\|\,\|f\|^4\nabla^{m-1}A\|^4_{L^\infty_\mu([\gamma>0])})\int_\Sigma |\nabla^m A|^2\gamma^s d\mu$$
$$+ c\|f\|^6_{L^\infty_\mu([\gamma>0])}\|A\|^2_{L^\infty_\mu([\gamma>0])}\int_{[\gamma>0]}|\nabla^2 A|^2 d\mu.$$

For the fourth term we use Young's inequality to obtain

$$\int_\Sigma \|f\|^7|\nabla A|^2|\nabla^{m-1}A||\nabla^m A|\gamma^s d\mu$$
$$\leq c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu + c\|\,\|f\|^4\nabla A\|^2_{L^\infty_\mu([\gamma>0])}\int_\Sigma \|f\|^6|\nabla A|^2|\nabla^{m-1}A|^2\gamma^s d\mu.$$

To estimate the second summand above we use integration by parts to derive

$$\int_\Sigma \|f\|^6|\nabla A|^2|\nabla^{m-1}A|^2\gamma^s d\mu$$
$$\leq c\int_\Sigma \|f\|^5|A||\nabla A||\nabla^{m-1}A|^2\gamma^s d\mu + c\int_\Sigma \|f\|^6|A||\nabla^2 A||\nabla^{m-1}A|^2\gamma^s d\mu$$
$$+ c\int_\Sigma \|f\|^6|A||\nabla A||\nabla^{m-1}A||\nabla^m A|\gamma^s d\mu$$
$$+ cs\Lambda\int_\Sigma \|f\|^6|A||\nabla A||\nabla^{m-1}A|^2\gamma^{s-1} d\mu$$
$$\leq \varepsilon\int_\Sigma \|f\|^6|\nabla A|^2|\nabla^{m-1}A|^2\gamma^s d\mu + c(\varepsilon)\int_{[\gamma>0]}\|f\|^4|A|^2|\nabla^{m-1}A|^2 d\mu$$
$$+ c\int_{[\gamma>0]}\|f\|^8|\nabla^2 A|^2|\nabla^{m-1}A|^2 d\mu + c(\varepsilon)\int_\Sigma \|f\|^6|A|^2|\nabla^m A|^2\gamma^s d\mu$$
$$+ c(\varepsilon)s^2\Lambda^2\int_{[\gamma>0]}\|f\|^6|A|^2|\nabla^{m-1}A|^2 d\mu,$$

i.e. by absorption and $m \geq 3$

$$\int_\Sigma \|f\|^6 |\nabla A|^2 |\nabla^{m-1} A|^2 \gamma^s d\mu$$
$$\leq c_m + c_m \| \, \|f\|^4 \nabla^{m-1} A\|^2_{L^\infty_\mu([\gamma>0])} + c_m \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu.$$

Inserting proves a correct estimate.
For the fifth term to be considered we clearly have

$$\int_\Sigma \|f\|^7 |A| |\nabla^3 A| |\nabla^{m-2} A| |\nabla^m A| \gamma^s d\mu$$
$$\leq c_m (1 + \| \, \|f\|^4 \nabla^{m-2} A\|^4_{L^\infty_\mu([\gamma>0])}) \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu + c \int_\Sigma |\nabla^3 A|^2 \gamma^s d\mu.$$

For the sixth and last one we estimate

$$\int_\Sigma \|f\|^7 |\nabla A| |\nabla^2 A| |\nabla^{m-2} A| |\nabla^m A| \gamma^s d\mu$$
$$\leq c \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu + c \| \, \|f\|^4 \nabla^2 A\|^2_{L^\infty_\mu([\gamma>0])} \int_\Sigma \|f\|^6 |\nabla A|^2 |\nabla^{m-2} A|^2 \gamma^s d\mu.$$

As above with $m-2$ instead of $m-1$ we obtain by integration by parts

$$\int_\Sigma \|f\|^6 |\nabla A|^2 |\nabla^{m-2} A|^2 \gamma^s d\mu$$
$$\leq c_m + c_m \| \, \|f\|^4 \nabla^{m-2} A\|^2_{L^\infty_\mu([\gamma>0])} + c_m \int_{[\gamma>0]} |\nabla^{m-1} A|^2 d\mu.$$

Inserting yields the required result.

- $k = 3, j = m, i = 2$

Clearly

$$\int_\Sigma (\|f\|^6 + \|f\|^7 |A|) |A|^2 |\nabla^m A|^2 \gamma^s d\mu \leq c_m \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu.$$

Next

$$\int_\Sigma (\|f\|^6 + \|f\|^7 |A|) |A| |\nabla A| |\nabla^{m-1} A| |\nabla^m A| \gamma^s d\mu$$
$$\leq c \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu + c_m (1 + \| \, \|f\|^4 \nabla^{m-1} A\|^4_{L^\infty_\mu([\gamma>0])}) \int_{[\gamma>0]} |\nabla A|^2 d\mu.$$

Likewise

$$\begin{aligned}\int_\Sigma &(\|f\|^6 + \|f\|^7|A|)|A||\nabla^2 A||\nabla^{m-2}A||\nabla^m A|\gamma^s d\mu\\ &\le c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu + c_m(1+\| \, \|f\|^4\nabla^{m-2}A\|^4_{L^\infty_\mu([\gamma>0])})\int_{[\gamma>0]}|\nabla^2 A|^2 d\mu.\end{aligned}$$

Finally we have

$$\begin{aligned}\int_\Sigma &(\|f\|^6 + \|f\|^7|A|)|\nabla A|^2|\nabla^{m-2}A||\nabla^m A|\gamma^s d\mu\\ &\le c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu + c_m(1+\|\|f\|^4\nabla^{m-2}A\|^4_{L^\infty_\mu([\gamma>0])})\int_\Sigma |\nabla A|^4\gamma^s d\mu\end{aligned}$$

and by integration by parts

$$\begin{aligned}\int_\Sigma |\nabla A|^4\gamma^s d\mu \le& c\int_\Sigma |A||\nabla A|^2|\nabla^2 A|\gamma^s d\mu + cs\Lambda\int_\Sigma |A||\nabla A|^3\gamma^{s-1}d\mu\\ \le&\varepsilon\int_\Sigma |\nabla A|^4\gamma^s d\mu + c(\varepsilon, c_m),\end{aligned}$$

i.e. by absorption

$$\sup_{0\le t<T}\int_\Sigma |\nabla A|^4\gamma^s d\mu \le c_m. \tag{10.2}$$

- $k = 3, j = m-1, i = 3$

$$\begin{aligned}\int_\Sigma &(\|f\|^5 + \|f\|^6|A| + \|f\|^7|A|^2 + \|f\|^7|\nabla A|)|A|^2|\nabla^{m-1}A||\nabla^m A|\gamma^s d\mu\\ &\le c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu + c_m(1+\| \, \|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])})\int_{[\gamma>0]}|\nabla^{m-1}A|^2 d\mu,\end{aligned}$$

and by (10.2)

$$\begin{aligned}\int_\Sigma &(\|f\|^5 + \|f\|^6|A| + \|f\|^7|A|^2 + \|f\|^7|\nabla A|)|A||\nabla A||\nabla^{m-2}A|\nabla^m A|\gamma^s d\mu\\ &\le c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu\\ &\quad + c_m(1+\| \, \|f\|^4\nabla^{m-2}A\|^4_{L^\infty_\mu([\gamma>0])})(\int_{[\gamma>0]}|\nabla A|^2 d\mu + \int_\Sigma |\nabla A|^4\gamma^s d\mu)\\ &\le c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu\\ &\quad + c_m(1+\| \, \|f\|^4\nabla^{m-2}A\|^4_{L^\infty_\mu([\gamma>0])}),\end{aligned}$$

- $k = 3, j = m-2, i = 4$

Clearly

$$\begin{aligned}\int_\Sigma (&\|f\|^4 + \|f\|^5|A| + \|f\|^6|A|^2 + \|f\|^6|\nabla A| \\ &+ \|f\|^7|A|^3 + \|f\|^7|A||\nabla A| + \|f\|^7|\nabla^2 A|)|A|^2|\nabla^{m-2}A||\nabla^m A|\gamma^s d\mu \\ \le& c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu + c_m(1 + \| \,\|f\|^4\nabla^{m-2}A\|^4_{L^\infty_\mu([\gamma>0])}).\end{aligned}$$

- $k = 3, j = 0, i = m+2$

It suffices to estimate

$$\begin{aligned}\int_\Sigma (\,&\|f\|^7|\nabla^m A| + \|f\|^7|A||\nabla^{m-1}A| + \|f\|^7[\,|\nabla A| + |A|^2\,]|\nabla^{m-2}A| \\ &+ \|f\|^6|\nabla^{m-1}A| + \|f\|^6|A||\nabla^{m-2}A| \\ &+ \|f\|^5|\nabla^{m-2}A|\,)\,|A|^3|\nabla^m A|\gamma^s d\mu \\ \le& c_m + c_m(1 + \| \,\|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])})\int_\Sigma |\nabla^m A|^2\gamma^s d\mu.\end{aligned}$$

- $k = 3, j = 1, i = m+1$

Similarly

$$\begin{aligned}\int_\Sigma &(\|f\|^7|\nabla^{m-1}A| + \|f\|^7|A||\nabla^{m-2}A| + \|f\|^6|\nabla^{m-2}A|)|A|^2|\nabla A||\nabla^m A|\gamma^s d\mu \\ \le& c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu \\ &+ c_m(1 + \| \,\|f\|^4\nabla^{m-2}A\|^4_{L^\infty_\mu([\gamma>0])} + \| \,\|f\|^4\nabla^{m-1}A\|^4_{L^\infty_\mu([\gamma>0])}).\end{aligned}$$

- $k = 3, j = 2, i = m$

By Young's inequality and (10.2) we derive

$$\begin{aligned}\int_\Sigma &\|f\|^7|\nabla^{m-2}A|(|A|^2|\nabla^2 A| + |A||\nabla A|^2)|\nabla^m A|\gamma^s d\mu \\ \le& c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu \\ &+ c_m(1 + \| \,\|f\|^4\nabla^{m-2}A\|^4_{L^\infty_\mu([\gamma>0])})(\int_{[\gamma>0]} |\nabla^2 A|^2 d\mu + \int_\Sigma |\nabla A|^4\gamma^s d\mu.) \\ \le& c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu + c_m(1 + \| \,\|f\|^4\nabla^{m-2}A\|^4_{L^\infty_\mu([\gamma>0])}).\end{aligned}$$

- $k = 1, j = m + 3, i = 1$

First we have by integration by parts

$$\begin{aligned}
&\int_\Sigma \nabla \|f\|^8 * \nabla^{m+3} A * \nabla^m A \gamma^s d\mu \\
&\quad \leq c(n) \int_\Sigma (\|f\|^6 + \|f\|^7 |A|) |\nabla^{m+2} A| |\nabla^m A| \gamma^s d\mu \\
&\qquad + c(n) \int_\Sigma \|f\|^7 |\nabla^{m+2} A| |\nabla^{m+1} A| \gamma^s d\mu \\
&\qquad + c(n) s\Lambda \int_\Sigma \|f\|^7 |\nabla^{m+2} A| |\nabla^m A| \gamma^{s-1} d\mu \\
&\quad \leq \varepsilon \int_\Sigma \|f\|^8 |\nabla^{m+2} A|^2 \gamma^s d\mu + c(\varepsilon, c_m) \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu \\
&\qquad + c(n, \varepsilon) \int_\Sigma \|f\|^6 |\nabla^{m+1} A|^2 \gamma^s d\mu + c(\varepsilon, c_m) \int_{[\gamma > 0]} \|f\|^8 |\nabla^m A|^2 d\mu.
\end{aligned}$$

Corollary 13.11 shows

$$\begin{aligned}
\int_\Sigma \|f\|^6 |\nabla^{m+1} A|^2 \gamma^s d\mu \leq{}& \varepsilon \int_\Sigma \|f\|^8 |\nabla^{m+2} A|^2 \gamma^s d\mu + c(\varepsilon) \int_\Sigma \|f\|^4 |\nabla^m A|^2 \gamma^s d\mu \\
&+ c(\varepsilon) s^2 \Lambda^2 \int_\Sigma \|f\|^6 |\nabla^m A|^2 \gamma^{s-2} d\mu \\
\leq{}& \varepsilon \int_\Sigma \|f\|^8 |\nabla^{m+2} A|^2 \gamma^s d\mu + c(\varepsilon, c_m) \int_\Sigma |\nabla^m A|^2 \gamma^s d\mu \\
&+ c(\varepsilon, c_m) \int_{[\gamma > 0]} \|f\|^8 |\nabla^m A|^2 d\mu. \qquad (10.3)
\end{aligned}$$

Inserting we conclude

$$\begin{aligned}\int_\Sigma &\nabla\|f\|^8 * \nabla^{m+3}A * \nabla^m A\gamma^s d\mu \\ &\le \varepsilon \int_\Sigma \|f\|^8 |\nabla^{m+2}A|\gamma^s d\mu + c(\varepsilon, c_m)\int_\Sigma |\nabla^m A|^2\gamma^s d\mu \\ &\quad + c(\varepsilon, c_m)\int_{[\gamma>0]} \|f\|^8|\nabla^m A|^2 d\mu,\end{aligned}$$

which is the required result.

- $k = 1, j = m + 2, i = 2$

We simply have

$$\begin{aligned}\int_\Sigma &(\|f\|^6 + \|f\|^7|A|)|\nabla^{m+2}A||\nabla^m A|\gamma^s d\mu \\ &\le \varepsilon \int_\Sigma \|f\|^8|\nabla^{m+2}A|\gamma^s d\mu + c(\varepsilon, c_m)\int_\Sigma |\nabla^m A|^2\gamma^s d\mu.\end{aligned}$$

- $k = 1, j = m + 1, i = 3$

Apply (10.3) to

$$\begin{aligned}\int_\Sigma &(\|f\|^5 + \|f\|^6|A| + \|f\|^7|A|^2 + \|f\|^7|\nabla A|)|\nabla^{m+1}A||\nabla^m A|\gamma^s d\mu \\ &\le c\int_\Sigma \|f\|^6|\nabla^{m+1}A|^2\gamma^s d\mu \\ &\quad + c_m(1 + \| \, \|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])})\int_\Sigma |\nabla^m A|^2\gamma^s d\mu.\end{aligned}$$

- $k = 1, j = m, i = 4$

Clearly

$$\begin{aligned}\int_\Sigma &(\|f\|^4 + \|f\|^5|A| + \|f\|^6|A|^2 + \|f\|^6|\nabla A| \\ &\quad + \|f\|^7|A|^3 + \|f\|^7|A||\nabla A| + \|f\|^7|\nabla^2 A|)|\nabla^m A|^2\gamma^s d\mu \\ &\le c_m(1 + \| \, \|f\|^4\nabla A\|^4_{L^\infty_\mu([\gamma>0])} + \| \, \|f\|^4\nabla^2 A\|^4_{L^\infty_\mu([\gamma>0])})\int_\Sigma |\nabla^m A|^2\gamma^s d\mu.\end{aligned}$$

- $k=1, j=m-1, i=5$

Since $m \geq 3$ and by (10.2) we have

$$\begin{aligned}
&\int_\Sigma (\|f\|^3 + \|f\|^4|A| + \|f\|^5|A|^2 + \|f\|^5|\nabla A| \\
&\qquad + \|f\|^6|A|^3 + \|f\|^6|A||\nabla A| + \|f\|^6|\nabla^2 A| \\
&\qquad + \|f\|^7|A|^4 + \|f\|^7|A|^2|\nabla A| + \|f\|^7|\nabla A|^2 \\
&\qquad + \|f\|^7|A||\nabla^2 A| + \|f\|^7|\nabla^3 A|)|\nabla^{m-1}A||\nabla^m A|\gamma^s d\mu \\
&\leq c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu + c_m\int_{[\gamma>0]}|\nabla^{m-1}A|^2 d\mu \\
&\quad + c_m(1 + \|\,\|f\|^4\nabla^{m-1}A\|^4_{L^\infty_\mu([\gamma>0])}) \\
&\qquad (\int_{[\gamma>0]}|\nabla A|^2 d\mu + \int_{[\gamma>0]}|\nabla^2 A|^2 d\mu \\
&\qquad\qquad + \int_\Sigma |\nabla A|^4\gamma^s d\mu + \int_\Sigma |\nabla^3 A|^2\gamma^s d\mu) \\
&\leq c_m(1 + \|\,\|f\|^4\nabla^{m-1}A\|^4_{L^\infty_\mu([\gamma>0])})(1 + \int_\Sigma |\nabla^m A|^2\gamma^s d\mu).
\end{aligned}$$

- $k=1, j=m-2, i=6$

Recalling $i \leq m+2$ from (4.6) we may assume $m \geq 4$ for this case. Hence

$$\begin{aligned}
&\int_\Sigma (\|f\|^2 + \|f\|^3|A| + \|f\|^4|A|^2 + \|f\|^4|\nabla A| \\
&\quad + \|f\|^5|A|^3 + \|f\|^5|A||\nabla A| + \|f\|^5|\nabla^2 A| \\
&\quad + \|f\|^6|A|^4 + \|f\|^6|A|^2|\nabla A| + \|f\|^6|\nabla A|^2 \\
&\quad + \|f\|^6|A||\nabla^2 A| + \|f\|^6|\nabla^3 A| \\
&\quad + \|f\|^7|A|^5 + \|f\|^7|A|^3|\nabla A| + \|f\|^7|A||\nabla A|^2 + \|f\|^7|A|^2|\nabla^2 A| \\
&\quad + \|f\|^7|\nabla A||\nabla^2 A| + \|f\|^7|A||\nabla^3 A| + \|f\|^7|\nabla^4 A|\ )\ |\nabla^{m-2}A||\nabla^m A|\gamma^s d\mu \\
&\leq c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu + c_m\int_{[\gamma>0]}|\nabla^{m-2}A|^2\gamma^s d\mu \\
&\quad + c_m(1 + \|\,\|f\|^4\nabla^{m-2}A\|^4_{L^\infty_\mu([\gamma>0])}) \\
&\qquad (\int_{[\gamma>0]}|\nabla^2 A|^2 d\mu + \int_{[\gamma>0]}|\nabla^3 A|^2 d\mu + \int_\Sigma |\nabla^4 A|^2\gamma^s d\mu) \\
&\leq c_m(1 + \|\,\|f\|^4\nabla^{m-2}A\|^4_{L^\infty_\mu([\gamma>0])})(1 + \int_\Sigma |\nabla^m A|^2\gamma^s d\mu),
\end{aligned}$$

where $\sup_{0\le t<T}\|\nabla A\|_{L^\infty_\mu([\gamma>0])}$, $\sup_{0\le t<T}\int_{[\gamma>0]}|\nabla^3 A|^2 d\mu \le d$ was used.

- $k=1, j=2, i=m+2$

Here we simply estimate by Young's inequality

$$\begin{aligned}
&\int_\Sigma (\|f\|^7|\nabla^m A| + \|f\|^7|A||\nabla^{m-1}A| + \|f\|^7(|A|^2+|\nabla A|)|\nabla^{m-2}A| \\
&\quad + \|f\|^6|\nabla^{m-1}A| + \|f\|^6|A||\nabla^{m-2}A| + \|f\|^5|\nabla^{m-2}A|\ )|\nabla^2 A||\nabla^m A|\gamma^s d\mu \\
&\le c\int_\Sigma \|f\|^6|\nabla A|^2|\nabla^{m-2}A|^2\gamma^s d\mu \\
&\quad + c_m + c_m(1+\|\ \|f\|^4\nabla^2 A\|^4_{L^\infty_\mu([\gamma>0])})\int_\Sigma |\nabla^m A|^2\gamma^s d\mu.
\end{aligned}$$

If $m=3$, apply (10.2). If $m\ge 4$, then $\int_\Sigma \|f\|^6|\nabla A|^2|\nabla^{m-2}A|^2\gamma^s d\mu \le c_m$, since $\sup_{0\le t<T}\|\nabla A\|_{L^\infty_\mu([\gamma>0])}\le d$ by assumption.

- $k=1, j=3, i=m+1$

Similarly

$$\begin{aligned}
&\int_\Sigma (\|f\|^7|\nabla^{m-1}A| + \|f\|^7|A||\nabla^{m-2}A| + \|f\|^6|\nabla^{m-2}A|)|\nabla^3 A||\nabla^m A|\gamma^s d\mu \\
&\qquad \le c\int_\Sigma |\nabla^m A|^2\gamma^s d\mu \\
&\qquad\quad + c_m(1+\|\ \|f\|^4\nabla^{m-2}A\|^4_{L^\infty_\mu([\gamma>0])} + \|\ \|f\|^4\nabla^{m-1}A\|^4_{L^\infty_\mu([\gamma>0])}) \\
&\qquad\qquad \int_\Sigma |\nabla^3 A|^2\gamma^s d\mu.
\end{aligned}$$

Recall $m\ge 3$.

- $k=1, j=4, i=m$

Finally, and this time really finally,

$$\begin{aligned}
&\int_\Sigma \|f\|^7|\nabla^{m-2}A||\nabla^4 A||\nabla^m A|\gamma^s d\mu \\
&\quad \le c\int_\Sigma \|f\|^6|\nabla^4 A|^2\gamma^s d\mu + c(1+\|\ \|f\|^4\nabla^{m-2}A\|^4_{L^\infty_\mu([\gamma>0])})\int_\Sigma |\nabla^m A|^2\gamma^s d\mu.
\end{aligned}$$

If $m\ge 4$, we have $\int_\Sigma \|f\|^6|\nabla^4 A|^2\gamma^s d\mu \le c_m + c_m\int_\Sigma |\nabla^m A|^2\gamma^s d\mu$.
If $m=3$, then $4=m+1$ and (10.3) yields a correct result. $\square$

11 Interpolation inequalities

We come to prove some interpolation inequalities, which will enable us to join the previous four sections to an induction.

Lemma 11.1. *For $k \in \{0, \mathbb{R}_{\geq 2}\}$, $s \geq 4$ and $\Lambda > 0$ there exists*

$$c = c(s, k, \Lambda) > 0,$$

such that, if $f : \Sigma \to \mathbb{R}^n$ is a closed, immersed surface, Φ a normal valued l-linear form along f and γ as in (5.11), we have

$$\begin{aligned}
&\| \, \|f\|^k \Phi\|^4_{L^\infty_\mu([\gamma=1])} \\
&\quad \leq c \int_{[\gamma>0]} \|f\|^{2k} |\Phi|^2 d\mu \, (\int_\Sigma \|f\|^{2k} |\nabla^2 \Phi|^2 \gamma^s d\mu + \int_\Sigma \|f\|^{2k} |A|^4 |\Phi|^2 \gamma^s d\mu \\
&\qquad + \int_{[\gamma>0]} [\, \|f\|^{2k} + k \, \|f\|^{2k-4} \,] |\Phi|^2 d\mu \,). \qquad (11.1)
\end{aligned}$$

Proof. We define $\Psi := \|f\|^k |\Phi| \gamma^{\frac{s}{2}}$.
Applying lemma 13.9 with $m = 2, p = 4, \alpha = \frac{2}{3}$ we obtain

$$\begin{aligned}
\|\Psi\|_{L^\infty_\mu(\Sigma)} &\leq c \|\Psi\|^{\frac{1}{3}}_{L^2_\mu(\Sigma)} (\|\nabla \Psi\|_{L^4_\mu(\Sigma)} + \|H\Psi\|_{L^4_\mu(\Sigma)})^{\frac{2}{3}} \\
&\leq c \|\Psi\|^{\frac{1}{3}}_{L^2_\mu(\Sigma)} (\|\Psi\|^{\frac{1}{3}}_{L^\infty_\mu(\Sigma)} \|\nabla^2 \Psi\|^{\frac{1}{3}}_{L^2_\mu(\Sigma)} + \|H\Psi\|^{\frac{2}{3}}_{L^4_\mu(\Sigma)}),
\end{aligned}$$

where proposition 13.10 for

$$\gamma = 1 \in C^\infty_0(\Sigma), \; k = u_1 = u_2 = v_1 = v_2 = \Lambda = 0 \text{ and } r = 2, \, p \to \infty$$

was used in the last step.
Since $\|H\Psi\|^{\frac{2}{3}}_{L^4_\mu(\Sigma)} \leq \|\Psi\|^{\frac{1}{3}}_{L^\infty_\mu(\Sigma)} \| \, |\Psi|^{\frac{1}{2}} |H| \, \|^{\frac{2}{3}}_{L^4_\mu(\Sigma)}$ we conclude

$$\|\Psi\|^4_{L^\infty_\mu(\Sigma)} \leq c \|\Psi\|^2_{L^2_\mu(\Sigma)} (\, \|\nabla^2 \Psi\|^2_{L^2_\mu(\Sigma)} + \| \, |\Psi|^2 |H|^4 \|_{L^1_\mu(\Sigma)} \,).$$

Recalling (5.11) we have for $k \geq 2$ and $c = c(s,k,\Lambda) > 0$

$$
\begin{aligned}
\|\nabla^2 \Psi\|^2_{L^2_\mu(\Sigma)} =& \int_\Sigma |\nabla^2(\|f\|^k \Phi \gamma^{\frac{s}{2}})|^2 d\mu \\
\leq& c \int_{[\gamma>0]} |\, (\|f\|^{k-2} + \|f\|^{k-1}|A|)|\Phi|\gamma^{\frac{s}{2}} + \|f\|^{k-1}|\nabla\Phi|\gamma^{\frac{s}{2}} \\
&\qquad + \|f\|^{k-1}|\Phi|\gamma^{\frac{s-2}{2}} + \|f\|^k|\nabla^2\Phi|\gamma^{\frac{s}{2}} \\
&\qquad + \|f\|^k|\nabla\Phi|\gamma^{\frac{s-2}{2}} + \|f\|^k|\Phi|(1+|A|\gamma)\gamma^{\frac{s-4}{2}}\ |^2 d\mu \\
\leq& c \int_\Sigma \|f\|^{2k}|\nabla^2\Phi|^2\gamma^s d\mu + c\int_\Sigma (\|f\|^{2k-2}\gamma^s + \|f\|^{2k}\gamma^{s-2})|\nabla\Phi|^2 d\mu \\
&+ c\int_{[\gamma>0]} [\ \|f\|^{2k-4}\gamma^s + \|f\|^{2k-2}\gamma^s|A|^2 \\
&\qquad + \|f\|^{2k-2}\gamma^{s-2} + \|f\|^{2k}\gamma^{s-4} + \|f\|^{2k}\gamma^{s-2}|A|^2\ ]|\Phi|^2 d\mu.
\end{aligned}
$$

By integration by parts

$$
\begin{aligned}
&\int_\Sigma [\ \|f\|^{2k-2}\gamma^s + \|f\|^{2k}\gamma^{s-2}\ ]|\nabla\Phi|^2 d\mu \\
&\quad \leq c(s,k,\Lambda)\int_\Sigma [\ \|f\|^{2k-3}\gamma^s + \|f\|^{2k-2}\gamma^{s-1} \\
&\qquad\qquad + \|f\|^{2k-1}\gamma^{s-2} + \|f\|^{2k}\gamma^{s-3}\ ]|\Phi||\nabla\Phi| d\mu \\
&\qquad + c\int_\Sigma [\ \|f\|^{2k-2}\gamma^s + \|f\|^{2k}\gamma^{s-2}\ ]|\Phi||\nabla^2\Phi| d\mu \\
&\quad \leq \varepsilon \int_\Sigma [\ \|f\|^{2k-2}\gamma^s + \|f\|^{2k}\gamma^{s-2}\ ]|\nabla\Phi|^2 d\mu + \int_\Sigma \|f\|^{2k}|\nabla^2\Phi|^2\gamma^s d\mu \\
&\qquad + c(\varepsilon,k,s,\Lambda)\int_{[\gamma>0]} [\ \|f\|^{2k-4}\gamma^s + \|f\|^{2k}\gamma^{s-4}\ ]|\Phi|^2 d\mu.
\end{aligned}
$$

Absorbing and plugging in we derive for $c = c(s,k,\Lambda) > 0$

$$
\begin{aligned}
\|\nabla^2\Psi\|^2_{L^2_\mu(\Sigma)} \leq& c\int_\Sigma \|f\|^{2k}|\nabla^2\Phi|^2\gamma^s d\mu + c\int_{[\gamma>0]} [\ \|f\|^{2k} + \|f\|^{2k-4}\gamma^4]|\Phi|^2\gamma^{s-4} d\mu \\
&+ c\int_\Sigma [\ \|f\|^{2k}\gamma^{s-2} + \|f\|^{2k-2}\gamma^s\ ]|A|^2|\Phi|^2 d\mu.
\end{aligned}
$$

Collecting terms we conclude

$$
\begin{aligned}
\| \, \|f\|^k \Phi\|^4_{L^\infty_\mu([\gamma=1])} \leq & \|\Psi\|^4_{L^\infty_\mu(\Sigma)} \\
\leq & c \| \, \|f\|^k \Phi \gamma^{\frac{s}{2}} \|^2_{L^2_\mu(\Sigma)} \\
& \qquad (\int_\Sigma \|f\|^{2k} |\nabla^2 \Phi|^2 \gamma^s d\mu \\
& \qquad + \int_{[\gamma>0]} [\, \|f\|^{2k} + \|f\|^{2k-4} \gamma^4 \,] |\Phi|^2 \gamma^{s-4} d\mu \\
& \qquad + \int_\Sigma [\, \|f\|^{2k} \gamma^{s-2} + \|f\|^{2k-2} \gamma^s \,] |A|^2 |\Phi|^2 d\mu \\
& \qquad + \int_\Sigma \|f\|^{2k} \gamma^s |H|^4 |\Phi|^2 d\mu \,) \\
\leq & c \int_{[\gamma>0]} \|f\|^{2k} |\Phi|^2 d\mu \, (\int_\Sigma \|f\|^{2k} |\nabla^2 \Phi|^2 \gamma^s d\mu \\
& \qquad + \int_{[\gamma>0]} [\, \|f\|^{2k} + \|f\|^{2k-4} \,] |\Phi|^2 d\mu \\
& \qquad + \int_\Sigma \|f\|^{2k} |A|^4 |\Phi|^2 \gamma^s d\mu \,).
\end{aligned}
$$

The case $k=0$ follows analogously. $\square$

Proposition 11.2. *For $n, m, \Lambda, R, d > 0$ and $s \geq 4$ there exist*

$$\varepsilon_3, \; c_3 = c_3(m, s, \Lambda, R, d) > 0$$

such that, if $f : \Sigma \to \mathbb{R}^n$ is a closed immersed surface, γ as in (5.11) and

$$
\begin{aligned}
& \int_{[\gamma>0]} |A|^2 d\mu \leq \varepsilon_3, \\
& \|f\|_{L^\infty_\mu([\gamma>0])} \leq R, \\
& \|A\|_{W^{m,2}_\mu([\gamma>0])} \leq d,
\end{aligned}
$$

we have for $k = 0, 4$

$$\| \, \|f\|^k \nabla^m A\|^4_{L^\infty_\mu([\gamma=1])} \leq c_3 (1 + \int_\Sigma \|f\|^{2k} |\nabla^{m+2} A|^2 \gamma^s d\mu).$$

Proof. Considering lemma 11.1 with $\Phi = \nabla^m A$ it suffices to estimate

$$\int_\Sigma \|f\|^{2k}|A|^4|\nabla^m A|^2\gamma^s d\mu \le c_3(1 + \int_\Sigma \|f\|^{2k}|\nabla^{m+2}A|^2\gamma^s d\mu).$$

For $m = k = 0$ we have by lemma 13.8

$$\begin{aligned}\int_\Sigma |A|^6\gamma^s d\mu =& \int_\Sigma (|A|^3\gamma^{\frac{s}{2}})^2 d\mu \\ \le& c[\int_\Sigma |A|^2|\nabla A|\gamma^{\frac{s}{2}} d\mu + s\Lambda \int_\Sigma |A|^3\gamma^{\frac{s-2}{2}} d\mu + \int |A|^4\gamma^{\frac{s}{2}} d\mu]^2 \\ \le& c\int_{[\gamma>0]} |A|^2 d\mu[\int_\Sigma |A|^2|\nabla A|^2\gamma^s d\mu + s^2\Lambda^2 \int_\Sigma |A|^4\gamma^{s-2} d\mu + \int_\Sigma |A|^6\gamma^s d\mu] \\ \le& c\int_\Sigma |A|^2|\nabla A|^2\gamma^s d\mu + c(\varepsilon, c_3) + c(\int_{[\gamma>0]} |A|^2 d\mu + \varepsilon)\int_\Sigma |A|^6\gamma^s d\mu.\end{aligned}$$

Absorbing yields

$$\int_\Sigma |A|^6\gamma^s d\mu \le c\int_\Sigma |A|^2|\nabla A|^2\gamma^s d\mu + c_3$$

Since $s \ge 4$, i.e. $\frac{s}{2} \le s - 2$, we have

$$\begin{aligned}\int_\Sigma |A|^2|\nabla A|^2\gamma^s d\mu =& \int_\Sigma (|A||\nabla A|\gamma^{\frac{s}{2}})^2 d\mu \\ \le& c[\int_\Sigma |\nabla A|^2\gamma^{\frac{s}{2}} d\mu + \int_\Sigma |A||\nabla^2 A|\gamma^{\frac{s}{2}} d\mu \\ &+ s\Lambda \int_\Sigma |A||\nabla A|\gamma^{\frac{s-2}{2}} d\mu + \int_\Sigma |A|^2|\nabla A|\gamma^{\frac{s}{2}} d\mu]^2 \\ \le& c\int_\Sigma |\nabla^2 A|^2\gamma^s d\mu + c_3 + c(\int_\Sigma |\nabla A|^2\gamma^{\frac{s}{2}} d\mu)^2 \\ &+ c\int_{[\gamma>0]} |A|^2 d\mu \int_\Sigma |A|^2|\nabla A|^2\gamma^s d\mu.\end{aligned}$$

Therefore by absorption

$$\begin{aligned}\int_\Sigma |A|^2|\nabla A|^2\gamma^s d\mu \le& c\int_\Sigma |\nabla^2 A|^2\gamma^s d\mu + c_3 + c(\int_\Sigma |\nabla A|^2\gamma^{\frac{s}{2}} d\mu)^2 \\ \le& c\int_\Sigma |\nabla^2 A|^2\gamma^s d\mu + c_3,\end{aligned}$$

where we used

$$
\begin{aligned}
(\int_\Sigma |\nabla A|^2 \gamma^{\frac{s}{2}} d\mu)^2 \leq & (c \int_\Sigma |A||\nabla^2 A| \gamma^{\frac{s}{2}} d\mu + cs\Lambda \int_\Sigma |A||\nabla A| \gamma^{\frac{s-2}{2}} d\mu)^2 \\
\leq & c \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu + c_3 \int_\Sigma |\nabla A|^2 \gamma^{s-2} d\mu \\
\leq & c \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu + \varepsilon (\int_\Sigma |\nabla A|^2 \gamma^{\frac{s}{2}} d\mu)^2 + c(\varepsilon, c_3)
\end{aligned}
$$

by integration by parts, Hölder's inequality and $\frac{s}{2} \leq s-2$.
We conclude for $\int_{[\gamma>0]} |A|^2 d\mu \leq \varepsilon_3$ sufficiently small

$$
\int_\Sigma |A|^6 \gamma^s \leq c_3 (1 + \int_\Sigma |\nabla^2 A|^2 \gamma^s d\mu).
$$

For $m = 0, k = 4$ we refer to (7.4).
For arbitrary $m \geq 1$ and $k = 4$ we estimate

$$
\begin{aligned}
\int_\Sigma \|f\|^8 |A|^4 |\nabla^m A|^2 \gamma^s d\mu = & \int_\Sigma (\|f\|^4 |A|^2 |\nabla^m A| \gamma^{\frac{s}{2}})^2 d\mu \\
\leq & c[\int_\Sigma \|f\|^3 |A|^2 |\nabla^m A| \gamma^{\frac{s}{2}} d\mu + \int_\Sigma \|f\|^4 |A||\nabla A||\nabla^m A| \gamma^{\frac{s}{2}} d\mu \\
& + \int_\Sigma \|f\|^4 |A|^2 |\nabla^{m+1} A| \gamma^{\frac{s}{2}} d\mu + s\Lambda \int_\Sigma \|f\|^4 |A|^2 |\nabla^m A| \gamma^{\frac{s-2}{2}} d\mu \\
& + \int_\Sigma \|f\|^4 |A|^3 |\nabla^m A| \gamma^{\frac{s}{2}} d\mu]^2 \\
\leq & c \int_{[\gamma>0]} |A|^2 d\mu \int_\Sigma \|f\|^8 |A|^2 |\nabla^{m+1} A|^2 \gamma^s d\mu \\
& + c(\int_{[\gamma>0]} |A|^2 d\mu + \varepsilon) \int_\Sigma \|f\|^8 |A|^4 |\nabla^m A|^2 \gamma^s d\mu \\
& + c(\varepsilon, c_3)((\int_{[\gamma>0]} |A|^2 d\mu)^2 + (\int_{[\gamma>0]} |\nabla A|^2 d\mu)^2) \\
& \int_{[\gamma>0]} [\|f\|^8 + \|f\|^4 \gamma^4] |\nabla^m A|^2 \gamma^{s-4} d\mu,
\end{aligned}
$$

i.e. by absorption

$$
\int_\Sigma \|f\|^8 |A|^4 |\nabla^m A|^2 \gamma^s d\mu \leq c \int_\Sigma \|f\|^8 |A|^2 |\nabla^{m+1} A|^2 \gamma^s d\mu + c_3.
$$

Since we have

$$
\begin{aligned}
\int_\Sigma \|f\|^8|A|^2|\nabla^{m+1}A|^2\gamma^s d\mu = & \int_\Sigma (\|f\|^4|A||\nabla^{m+1}A|\gamma^{\frac{s}{2}})^2 d\mu \\
\le & c[\int_\Sigma \|f\|^3|A||\nabla^{m+1}A|\gamma^{\frac{s}{2}} d\mu + \int_\Sigma \|f\|^4|\nabla A||\nabla^{m+1}A|\gamma^{\frac{s}{2}} d\mu \\
& + \int_\Sigma \|f\|^4|A||\nabla^{m+2}A|\gamma^{\frac{s}{2}} d\mu + s\Lambda\int_\Sigma \|f\|^4|A||\nabla^{m+1}A|\gamma^{\frac{s-2}{2}} d\mu \\
& + \int_\Sigma \|f\|^4|A|^2|\nabla^{m+1}A|\gamma^{\frac{s}{2}} d\mu]^2 \\
\le & c\int_{[\gamma>0]} |A|^2 d\mu \int_\Sigma \|f\|^8|\nabla^{m+2}A|^2\gamma^s d\mu \\
& + c\int_{[\gamma>0]} |A|^2 d\mu \int_\Sigma \|f\|^8|A|^2|\nabla^{m+1}A|^2\gamma^s d\mu \\
& + c_3(\int_{[\gamma>0]} |A|^2 d\mu + \int_{[\gamma>0]} |\nabla A|^2 d\mu) \\
& (\int_\Sigma \|f\|^8|\nabla^{m+1}A|^2\gamma^{s-2} d\mu + \int_\Sigma \|f\|^6|\nabla^{m+1}A|^2\gamma^s d\mu)
\end{aligned}
$$

and by corollary 13.11 for $p = 2, u = 0, v = 1$ and $p = 2, u = 1, v = 0$

$$
\begin{aligned}
\int_\Sigma \|f\|^8|\nabla^{m+1}A|^2\gamma^{s-2} d\mu + \int_\Sigma \|f\|^6|\nabla^{m+1}A|^2\gamma^s d\mu & \\
\le & \varepsilon\int_\Sigma \|f\|^8|\nabla^{m+2}A|^2\gamma^s d\mu \\
& + c(\varepsilon, s, \Lambda)\int_{[\gamma>0]} [\|f\|^8\gamma^{s-4} + \|f\|^4\gamma^s]|\nabla^m A|^2 d\mu,
\end{aligned}
$$

we conclude by absorption

$$
\int_\Sigma \|f\|^8|A|^2|\nabla^{m+1}A|^2\gamma^s d\mu \le c\int_\Sigma \|f\|^8|\nabla^{m+2}A|^2\gamma^s d\mu + c_3.
$$

Consequently for $\int_{[\gamma>0]} |A|^2 d\mu \le \varepsilon_3$ sufficiently small

$$
\int_\Sigma \|f\|^8|A|^4|\nabla^m A|^2\gamma^s d\mu \le c_3(1 + \int_\Sigma \|f\|^8|\nabla^{m+2}A|\gamma^s d\mu).
$$

For arbitrary $m \geq 1$ and $k = 0$ we have

$$\begin{aligned}
\int_\Sigma |A|^4 |\nabla^m A|^2 \gamma^s d\mu = & \int_\Sigma (|A|^2 |\nabla^m A| \gamma^{\frac{s}{2}})^2 d\mu \\
\leq & c[\int_\Sigma |A||\nabla A||\nabla^m A|\gamma^{\frac{s}{2}} d\mu + \int_\Sigma |A|^2 |\nabla^{m+1} A| \gamma^{\frac{s}{2}} d\mu \\
& + s\Lambda \int_\Sigma |A|^2 |\nabla^m A| \gamma^{\frac{s-2}{2}} d\mu + \int_\Sigma |A|^3 |\nabla^m A| \gamma^{\frac{s}{2}} d\mu]^2 \\
\leq & c \int_{[\gamma>0]} |A|^2 d\mu \int_\Sigma |A|^2 |\nabla^{m+1} A|^2 \gamma^s d\mu \\
& + c(\int_{[\gamma>0]} |A|^2 d\mu + \varepsilon) \int_\Sigma |A|^4 |\nabla^m A|^2 \gamma^s d\mu \\
& + c(\varepsilon, c_3)((\int_{[\gamma>0]} |A|^2 d\mu)^2 + (\int_{[\gamma>0]} |\nabla A|^2 d\mu)^2) \int_{[\gamma>0]} |\nabla^m A|^2 d\mu,
\end{aligned}$$

i.e. by absorption $\int_\Sigma |A|^4 |\nabla^m A|^2 \gamma^s d\mu \leq c \int_\Sigma |A|^2 |\nabla^{m+1} A|^2 \gamma^s d\mu + c_3$. Since

$$\begin{aligned}
\int_\Sigma |A|^2 |\nabla^{m+1} A|^2 \gamma^s d\mu = & \int_\Sigma (|A||\nabla^{m+1} A| \gamma^{\frac{s}{2}})^2 d\mu \\
\leq & c \int_{[\gamma>0]} |A|^2 d\mu \int_\Sigma |\nabla^{m+2} A|^2 \gamma^s d\mu + c \int_{[\gamma>0]} |A|^2 d\mu \int_\Sigma |A|^2 |\nabla^{m+1} A|^2 \gamma^s d\mu \\
& + c_3(\int_{[\gamma>0]} |A|^2 d\mu + \int_{[\gamma>0]} |\nabla A|^2 d\mu) \int_\Sigma |\nabla^{m+1} A|^2 \gamma^{s-2} d\mu
\end{aligned}$$

and by corollary 13.11

$$\int_\Sigma |\nabla^{m+1} A|^2 \gamma^{s-2} d\mu \leq \varepsilon \int_\Sigma |\nabla^{m+2} A|^2 \gamma^s d\mu + c(\varepsilon, c_3) \int_{[\gamma>0]} |\nabla^m A|^2 d\mu,$$

we obtain by absorption

$$\int_\Sigma |A|^2 |\nabla^{m+1} A|^2 \gamma^s d\mu \leq c \int_\Sigma |\nabla^{m+2} A|^2 \gamma^s d\mu + c_3$$

and consequently for $\int_{[\gamma>0]} |A|^2 d\mu \leq \varepsilon_3$

$$\int_\Sigma |A|^4 |\nabla^m A|^2 \gamma^s d\mu \leq c_3(1 + \int_\Sigma |\nabla^{m+2} A|^2 \gamma^s d\mu).$$

$\square$

12 The main results

Theorem 12.1. *For $n, m, R, \tau > 0$ and $\alpha \in \mathbb{R}^{m+2}_{>0}$ there exist*

$$\varepsilon = \varepsilon(n),\ c = c(n, m, R, \alpha, \tau) > 0$$

such that, if $\quad f : \Sigma \times [0, T) \longrightarrow \mathbb{R}^n \setminus \{\mathbb{0}\}, \quad 0 < T \leq \tau,$
is an inverse Willmore flow satisfying

$$\begin{aligned}
&\sup_{0 \leq t < T} \int_{B_{2\rho}(x_0)} |A|^2 d\mu \leq \varepsilon, \\
&\sup_{0 \leq t < T} \rho^{-1} \|f\|_{L^\infty_\mu(B_{2\rho}(x_0))} \leq R, \\
&\rho^{2i} \int_{B_{2\rho}(x_0)} |\nabla^i A|^2 d\mu \lfloor_{t=0} \leq \alpha_i, \quad i = 1, \ldots, m+2
\end{aligned}$$

for some $x_0 \in \mathbb{R}^n$ and $\rho > 0$, we have

$$\sup_{0 \leq t < T} \|\nabla^m A\|_{L^\infty_\mu(B_\rho(x_0))} \leq \rho^{-(m+1)} c.$$

Proof. Rescaling, cf. Lemmata 13.5 and 13.7, we may assume $\rho = 1$.
Define

$$\varepsilon(n) := \min\{\varepsilon_0(n), \varepsilon_1(n), \varepsilon_2(n), \varepsilon_3\},$$

cf. propositions 7.2, 8.1, 9.1 and 11.2 and choose a sequence $(\tilde{\gamma}_i) \subset C^2_0(\mathbb{R}^n)$, which satisfies

1. $1 \geq \tilde{\gamma}_0 \geq \tilde{\gamma}_1 \geq \ldots \geq \tilde{\gamma}_m \geq \ldots \geq 0$,
2. $B_2(x_0) \supset supp(\tilde{\gamma}_0) \supset [\tilde{\gamma}_0 = 1] \supset supp(\tilde{\gamma}_1) \supset [\tilde{\gamma}_1 = 1] \supset \ldots \supset B_1(x_0)$,
3. $\|\tilde{\gamma}_i\|_{C^2(\mathbb{R}^n)} \leq c_i$ for all $i \geq 0$.

Let $\gamma_i := \tilde{\gamma}_i \circ f$. From proposition 7.2 for $s = 4, \gamma = \gamma_0$ we infer

$$\int_0^T \int_\Sigma \|f\|^8 |\nabla^2 A|^2 \gamma_0^4 \, d\mu \, dt \leq c(n, R, \tau)$$

and hence by proposition 11.2 for $m = 0$, $k = s = 4$ and $d = \varepsilon$

$$\begin{aligned}
\int_0^T \| \, \|f\|^4 A\|^4_{L^\infty_\mu([\gamma_0 = 1])} \, dt \leq & c(n, R)(T + c(n, R, \tau)) \\
\leq & c(n, R, \tau).
\end{aligned}$$

Since $[\gamma_0 = 1] \supset supp(\gamma_1)$, we conclude

$$\int_0^T \int_{[\gamma_1>0]} \|f\|^8 |\nabla^2 A|^2 \, d\mu \, dt \leq c(n, R, \tau),$$
$$\int_0^T \| \, \|f\|^4 A\|^4_{L^\infty_\mu([\gamma_1>0])} \, dt \leq c(n, R, \tau).$$

Next we have by proposition 8.1 for $s = 6$, $\gamma = \gamma_1$ and $d = c(n, R, \tau)$

$$\begin{aligned}\sup_{0\leq t<T} \int_\Sigma |\nabla A|^2 \gamma_1^6 d\mu + \int_0^T \int_\Sigma \|f\|^8 |\nabla^3 A|^2 \gamma_1^6 d\mu \; dt \\ \leq & c(n, R, d, \tau)(1 + \int_\Sigma |\nabla A|^2 \gamma_1^6 d\mu \lfloor_{t=0}) \\ \leq & c(n, R, \alpha_1, \tau).\end{aligned}$$

By proposition 11.2 for $m = 1$, $k = 4$, $s = 6$ and $d = c(n, R, \alpha_1, \tau)$ and

$$[\gamma_1 = 1] \supset supp(\gamma_2) \supset [\gamma_2 = 1] \supset supp(\gamma_3)$$

we derive

$$\sup_{0\leq t<T} \int_{[\gamma_3>0]} |\nabla A|^2 d\mu \leq c(n, R, \alpha_1, \tau),$$
$$\int_0^T \int_{[\gamma_3>0]} \|f\|^8 |\nabla^{k+2} A|^2 \, d\mu \, dt \leq c(n, R, \alpha_1, \tau),$$
$$\int_0^T \| \, \|f\|^4 \nabla^k A\|^4_{L^\infty_\mu([\gamma_3>0])} \, dt \leq c(n, R, \alpha_1, \tau)$$

for $k = 0, 1$.
Hence by proposition 9.1 for $s = 8$, $\gamma = \gamma_3$ and $d = c(n, R, \alpha_1, \tau)$

$$\begin{aligned}\sup_{0\leq t<T} \int_\Sigma |\nabla^2 A|^2 \gamma_3^8 d\mu + \int_0^T \int_\Sigma \|f\|^8 |\nabla^4 A|^2 \gamma_3^8 d\mu \; dt \\ \leq & c(n, R, d, \tau)(1 + \int_\Sigma |\nabla^2 A|^2 \gamma_3^8 d\mu \lfloor_{t=0}) \\ \leq & c(n, R, \alpha_1, \alpha_2, \tau).\end{aligned}$$

By proposition 11.2 for $m = 0, 2, \; k = 0, 4, \; s = 8, \; d = c(n, R, \alpha_1, \alpha_2, \tau)$ and,

since

$$[\gamma_3 = 1] \supset supp(\gamma_4),$$

we obtain both

$$\sup_{0\le t<T} \|A\|_{L^\infty_\mu[(\gamma_4=1])} \le c(n,R,\alpha_1,\alpha_2,\tau),$$

$$\int_0^T \| \, \|f\|^4 \nabla^2 A\|^4_{L^\infty_\mu([\gamma_4=1])} \, dt \le c(n,R,\alpha_1,\alpha_2,\tau). \tag{12.1}$$

Therefore and by $[\gamma_4 = 1] \supset supp(\gamma_5)$ we may assume for

$$m \ge 3, \; \gamma = \gamma_{2m-1}$$

inductively

$$\begin{aligned}
&\sup_{0\le t<T} \|A\|_{W^{m-3,\infty}_\mu([\gamma_{2m-1}>0])} \le d,\\
&\int_0^T \| \, \|f\|^4 \nabla^k A\|^4_{L^\infty_\mu([\gamma_{2m-1}>0])} dt \le d \quad \text{for} \quad k = m-2, m-1,\\
&\sup_{0\le t<T} \|A\|_{W^{m-1,2}_\mu([\gamma_{2m-1}>0])} \le d,\\
&\int_0^T \int_{[\gamma_{2m-1}>0]} \|f\|^8 |\nabla^k A|^2 d\mu \, dt \le d \quad \text{for} \quad k = m, m+1.
\end{aligned}$$

with $d = c(n,m,R,\alpha_1,\ldots,\alpha_{m-1},\tau)$.
By proposition 10.1 for $s = 2m+4$ we conclude

$$\begin{aligned}
\sup_{0\le t<T} \int_\Sigma |\nabla^m A|^2 \gamma_{2m-1}^{2m+4} d\mu + \int_0^T \int_\Sigma \|f\|^8 |\nabla^{m+2} A|^2 \gamma_{2m-1}^{2m+4} d\mu \, dt&\\
\le & c(n,m,R,d,\tau)(1 + \int_\Sigma |\nabla^m A|^2 \gamma_{2m-1}^{2m+4} d\mu \lfloor_{t=0})\\
\le & c(n,m,R,\alpha_1,\ldots,\alpha_m,\tau).
\end{aligned}$$

Now the claim follows by induction over $m \ge 3$ from proposition 11.2 and

$$[\gamma_{2m-1} = 1] \supset supp(\gamma_{2m}) \supset [\gamma_{2m} = 1] \supset supp(\gamma_{2m+1}) = supp(\gamma_{2(m+1)-1}).$$

$\square$

Next we state a version of theorem 1.1.

Theorem 12.2. *For* $0 < R, \alpha_1, \alpha_2 < \infty$ *and* $n \in \mathbb{N}$ *there exist*

$$\delta = \delta(n),\ k = k(n),\ c = c(n, R, \alpha_1, \alpha_2) > 0$$

such that, if $f_0 : \Sigma \longrightarrow \mathbb{R}^n \setminus \{\mathbb{0}\}$ *is a closed immersed surface satisfying*

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n} \int_{B_{2\rho}(x)} |A|^2 d\mu \le \delta, \\
&\sup_{x\in\mathbb{R}^n} \rho^{2i} \int_{B_{2\rho}(x)} |\nabla^i A|^2 d\mu \le \alpha_i, \quad i = 1, 2\ , \\
&\rho^{-1} \|f_0\|_{L^\infty_\mu(\Sigma)} \le R
\end{aligned}$$

for some $\rho > 0$*, there exists an inverse Willmore flow*

$$f : \Sigma \times [0, T) \longrightarrow \mathbb{R}^n \setminus \{\mathbb{0}\}, \quad f(\cdot, 0) = f_0$$

with $T > \rho^{-4} c$*. Moreover we have the estimates*

$$\sup_{0\le t\le\rho^{-4}c} \rho^{-1}\|f\|_{L^\infty_\mu(\Sigma)} \le 2R \quad \textit{and} \quad \sup_{0\le t\le\rho^{-4}c,\, x\in\mathbb{R}^n} \int_{B_{2\rho}(x)} |A|^2 d\mu \le k\delta.$$

Proof. Rescaling we may assume $\rho = 1$. Let $T > 0$ be maximal with respect to the existence of an inverse Willmore flow

$$f : \Sigma \times [0, T) \longrightarrow \mathbb{R}^n \setminus \{\mathbb{0}\}, \quad f(\cdot, 0) = f_0$$

We clearly have for some $\Gamma = \Gamma(n) > 1$

$$\varepsilon(t) := \sup_{x\in\mathbb{R}^n} \int_{B_2(x)} |A|^2 d\mu \le \Gamma \sup_{x\in\mathbb{R}^n} \int_{B_1(x)} |A|^2 d\mu.$$

Let $\delta := \frac{\varepsilon}{3\Gamma}$, cf. theorem 8.1, and

$$t_b := \sup\{0 \le t < T \mid \sup_{0\le\tau<t} \|f\|_{L^\infty_\mu(\Sigma)} \le 2R\}.$$

For arbitrary $\lambda > 0$ we have by continuity

$$t_0 := \sup\{0 \le t < \min(T, \lambda, t_b) \mid \varepsilon(\tau) \le 3\Gamma\delta \text{ for all } 0 \le \tau < t\} > 0.$$

If $t_0 < \min(T, \lambda, t_b)$, then we have by definition of δ and t_b

$$\begin{aligned}
&\varepsilon(t_0) = 3\Gamma\delta = \varepsilon,\\
&\sup_{0\le t<t_0,\, x\in\mathbb{R}^n} \int_{B_2(x)} |A|^2 d\mu \le \varepsilon,\\
&\sup_{0\le t<t_0} \|f\|_{L^\infty_\mu(\Sigma)} \le 2R.
\end{aligned}$$

Hence we obtain for suitable

$$\widetilde{\gamma} \in C^2_0(B_2(0)) \quad \text{with} \quad \mathbb{1}_{B_1(0)} \le \widetilde{\gamma} \le \mathbb{1}_{B_2(0)} \quad \text{and} \quad \|\widetilde{\gamma}\|_{C^2(\mathbb{R}^n)} \le \Lambda = \Lambda(n),$$

by proposition 7.2 choosing $s = 4$ and $\widetilde{\gamma}_x := \widetilde{\gamma}(\cdot - x) \circ f$ for $x \in \mathbb{R}^n$

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} \int_{B_1(x)} |A|^2 d\mu\lfloor_{t=t_0} &\le \sup_{x\in\mathbb{R}^n} \int_{B_2(x)} |A|^2 d\mu\lfloor_{t=0} + c(n)((2R)^8 + (2R)^4)\, t_0\\
&\le \delta + c(n)(R^8 + R^4)\,\lambda \le 2\delta,
\end{aligned}$$

defining $\lambda := \frac{\varepsilon}{3\Gamma c(n)(R^8+R^4)}$. Hence $\varepsilon(t_0) \le 2\Gamma\delta < \varepsilon$, a contradiction, and

$$t_0 = \min(T, \frac{c(n)}{R^8 + R^4}, t_b).$$

We will show, that $t_0 = \frac{c(n)}{R^8+R^4}$ or $t_0 = t_b$ imply $T > c(n, R, \alpha_1, \alpha_2)$, whereas $t_0 = T$ leads to a contradiction, what proves the claim.
If $t_0 = \frac{c(n)}{R^8+R^4} < T$, then $T > c(n, R)$.
If $t_0 = t_b < T$, then we have

$$\begin{aligned}
&\sup_{0\le t<t_0,\, x\in\mathbb{R}^n} \int_{B_2(x)} |A|^2 d\mu \le \varepsilon,\\
&\sup_{0\le t<t_0} \|f\|_{L^\infty_\mu(\Sigma)} \le 2R,\\
&\sup_{x\in\mathbb{R}^n} \int_{B_2(x)} |\nabla^i A|^2 d\mu\lfloor_{t=0} \le \alpha_i,\ i = 1, 2\ ,\\
&\|f(\sigma, 0)\| \le R, \quad \|f(\sigma, t_0)\| = 2R
\end{aligned}$$

for some $\sigma \in \Sigma$. For $t_0 > 1$, we have $T > 1$. Otherwise we have $t_0 \le 1$ and infer from Theorem 8.1 for $m = 0$, $\rho = \tau = 1$, taking (12.1) into account

$$\sup_{0\le t<t_0} \|A\|_{L^\infty_\mu(\Sigma)},\ \int_0^{t_0} \|\, \|f\|^4 \nabla^2 A\|^4_{L^\infty_\mu(\Sigma)}\, dt \le c(n, R, \alpha_1, \alpha_2)$$

by covering $\overline{B_{2R}(0)} \subset \cup_{i \in N = N(n,R)} B_1(x_i)$. It follows

$$\begin{aligned}
2R =& \|f(\sigma, t_0)\| = \|f(\sigma, 0) + \int_0^{t_0} \partial_t f(\sigma, s) ds\| \\
\leq& R + c \int_0^{t_0} [\, \|f\|^8 (|\nabla^2 A| + |A|^3)](\sigma, s) ds \\
\leq& R + cR^4 \int_0^{t_0} \| \, \|f\|^4 \nabla^2 A \|_{L^\infty_\mu(\Sigma)} dt + cR^8 \int_0^{t_0} \|A\|^3_{L^\infty_\mu(\Sigma)} dt \\
\leq& R + cR^4 t_0^{\frac{3}{4}} [\int_0^{t_0} \| \, \|f\|^4 \nabla^2 A \|^4_{L^\infty_\mu(\Sigma)} dt]^{\frac{1}{4}} + c(n, R, \alpha_1, \alpha_2) t_0 \\
\leq& \frac{3}{2} R + c(n, R, \alpha_1, \alpha_2) t_0,
\end{aligned}$$

what shows

$$c(n, R, \alpha_1, \alpha_2) t_0 \geq R.$$

Hence

$$T > \min(1, \frac{R}{c(n, R, \alpha_1, \alpha_2)}) =: c(n, R, \alpha_1, \alpha_2).$$

If $t_0 = T$, then we have

$$\begin{aligned}
& t_0 \leq \frac{c(n)}{R^8 + R^4}, \\
& \sup_{0 \leq t < t_0,\, x \in \mathbb{R}^n} \int_{B_2(x)} |A|^2 d\mu \leq \varepsilon, \\
& \sup_{0 \leq t < t_0} \|f\|_{L^\infty_\mu(\Sigma)} \leq 2R
\end{aligned}$$

and hence by theorem 12.1 for $\rho = 1$, $\tau = \frac{c(n)}{R^8 + R^4}$, $\alpha_i := \int_\Sigma |\nabla^i A|^2 d\mu \lfloor_{t=0}$

$$\sup_{0 \leq t < t_0} \|\nabla^m A\|_{L^\infty_\mu(\Sigma)} \leq c(n, m, R, \alpha_0, \ldots, \alpha_{m+2}) < \infty. \tag{12.2}$$

From this we will draw several conclusions.

1. For the metric we derive by (2.7) and (3.2)

$$\partial_t g = -2\langle A, \partial_t f\rangle = \|f\|^8(\nabla^2 A * A + A * A * A * A)$$

   and hence

$$\sup_{0\le t<t_0} \|\partial_t g\|_{L^\infty_\mu(\Sigma)} < \infty.$$

   Particularly the metrics $g(t), t \in [0, t_0)$, are equivalent and converge uniformly to a metric $g(t) \longrightarrow: g(t_0)$ for $t \longrightarrow t_0$, cf. [3], Lemma 14.2.

2. Similarly we obtain for the Christoffel symbols, cf. (2.9)

$$\begin{aligned}\partial_t \Gamma^k_{i,j} = &- g^{l,k}(\langle \nabla_l A_{i,j}, \partial_t f\rangle - \langle A_{i,j}, \nabla_l \partial_t f\rangle \\ &+ \langle A_{i,l}, \nabla_j \partial_t f\rangle + \langle A_{j,l}, \nabla_i \partial_t f\rangle),\end{aligned}$$

   what shows

$$\partial_t \Gamma = \|f\|^8(P^3_2(A) + P^1_4(A)) + \nabla\|f\|^8 * (P^2_1(A) + P^0_4(A))$$

   and therefore by virtue of corollary 13.4

$$\sup_{0\le t<t_0} \|\nabla^m \partial_t \Gamma\|_{L^\infty_\mu(\Sigma)} < \infty. \tag{12.3}$$

3. Turning to the coordinate derivatives we note for $T \in T(p, q)$

$$\nabla^m T = \partial^m T + \sum_{l=1}^{m} \sum_{k_1+\ldots+k_l+k\le m-1} \partial^{k_1}\Gamma * \ldots * \partial^{k_l}\Gamma * \partial^k T,$$

   which is easily checked by induction. This shows

$$|\partial^m T| \le c(n,m)(1 + |\Gamma| + \ldots + |\partial^{m-1}\Gamma|)^m(|\nabla^m T| + |\partial^{m-1} T| + \ldots + |T|)$$

   and inductively

$$|\partial^m T| \le c(n,m)(1 + |\Gamma| + \ldots + |\partial^{m-1}\Gamma|)^{\sum_{i=1}^m i}(|\nabla^m T| + \ldots + |T|).$$

   For $T = \partial_t \Gamma$ we conclude by $\partial^m \partial_t \Gamma = \partial_t \partial^m \Gamma$ and (12.3) inductively

$$\sup_{0\le t<t_0} \|\partial^m \partial_t \Gamma\|_{L^\infty_\mu(\Sigma)}, \ \sup_{0\le t<t_0} \|\partial^m \Gamma\|_{L^\infty_\mu(\Sigma)} < \infty. \tag{12.4}$$

4. For $k+l=m\geq 0$ we have

$$\sup_{0\leq t<t_0}\|\partial^k\nabla^l A\|_{L^\infty_\mu(\Sigma)},\ \sup_{0\leq t<t_0}\|\partial^{m+1}f\|_{L^\infty_\mu(\Sigma)}<\infty. \tag{12.5}$$

This holds obviously true for $m=0$ by (12.2). Now for $k+l=m+1$

$$\begin{aligned}\partial^k\nabla^l A-\nabla^{m+1}A=&\partial^k\nabla^l A-\nabla^k\nabla^l A\\ =&\sum_{i=1}^{k}\partial^{i-1}(\partial-\nabla)\nabla^{m+1-i}A.\end{aligned}$$

Since for any normal valued l-linear form Φ along f

$$0=\partial_i\langle\partial_j f,\Phi_{i_1,\dots,i_l}\rangle=\langle A_{i,j},\Phi_{i_1,\dots,i_l}\rangle+\langle\partial_j f,\partial_i\Phi_{i_1,\dots,i_l}\rangle$$

and hence

$$\begin{aligned}\nabla_i\Phi_{i_1,\dots,i_l}=&\partial_i^\perp\Phi_{i_1,\dots,i_l}-\sum_{k=1}^{l}\Phi_{i_1,\dots,k,\dots,i_l}\Gamma^k_{i,i_k}\\ =&\partial_i\Phi_{i_1,\dots,i_l}+g^{k,j}\langle A_{i,j},\Phi_{i_1,\dots,i_l}\rangle\partial_k f-\sum_{k=1}^{l}\Phi_{i_1,\dots,k,\dots,i_l}\Gamma^k_{i,i_k},\end{aligned}$$

we derive

$$\partial^k\nabla^l A-\nabla^{m+1}A=\sum_{i=1}^{k}\partial^{i-1}(\nabla^{m+1-i}A*(A*\partial f+\Gamma)).$$

By induction hypothesis and (12.4)

$$\partial^k\nabla^l A \text{ for } k+l\leq m,\ \partial^i f \text{ for } 1\leq i\leq m+1 \text{ and } \partial^m\Gamma \text{ for all } m\in\mathbb{N}$$

are bounded. This proves

$$\sup_{0\leq t<t_0}\|\partial^k\nabla^l A\|_{L^\infty_\mu(\Sigma)}<\infty \quad\text{for}\quad k+l=m+1. \tag{12.6}$$

By $\partial_{i,j}f=A_{i,j}+\Gamma^k_{i,j}\partial_k f$ we then have

$$\partial^{m+2}f=\partial^m A+\sum_{k+l=m}\partial^k\Gamma\cdot\partial^{l+1}f.$$

The induction hypothesis, (12.4) and (12.6) yield

$$\sup_{0\leq t<t_0}\|\partial^{m+2}f\|_{L^\infty_\mu(\Sigma)}<\infty.$$

This proves (12.5).

From the evolution equation (3.2) of the inverse Willmore flow

$$\partial_t f = -\frac{\|f\|^8}{2}(\Delta H + Q(A^0)H)$$

and (12.5) we infer, since $\sup_{0\leq t<t_0}\|f\|_{L^\infty_\mu(\Sigma)} \leq 2R$, that for all $m \geq 0$

$$\sup_{0\leq t<t_0}\|\partial^m f\|_{L^\infty_\mu(\Sigma)}, \ \sup_{0\leq t<t_0}\|\partial^m \partial_t f\|_{L^\infty_\mu(\Sigma)} < \infty.$$

Therefore $f(t) \longrightarrow: f(t_0)$ in $C^m(\Sigma, \mathbb{R}^n)$ as $t \nearrow t_0$ for all $m \geq 0$.
We conclude by what was already proven above, see the first point,

$$g(t_0) \longleftarrow g(t) = g_{f(t)} \longrightarrow g_{f(t_0)} \quad \text{uniformly.}$$

Thus $g_{f(t_0)}$ is a metric and hence $f(t_0) : \Sigma \longrightarrow \mathbb{R}^n$ a smooth immersion. If $f(t_0) : \Sigma \longrightarrow \mathbb{R}^n \setminus \{\mathbb{0}\}$, we may extend the inverse Willmore flow by short time existence, what shows, that $t_0 = T$ could not have been the maximal time of lifespan of f. This contradicts the maximality of T.
Consequently it remains to show, that $f(t_0) : \Sigma \longrightarrow \mathbb{R}^n \setminus \{\mathbb{0}\}$.
Otherwise there exists by continuity $\sigma \in \Sigma$, such that

$$f(\sigma, t) \longrightarrow \mathbb{0} \text{ as } t \nearrow t_0.$$

We choose a sequence $\tau_n \nearrow t_0$ satisfying

$$0 \longleftarrow \|f(\sigma, \tau_n)\| \geq \|f(\sigma, t)\| \quad \text{for all} \quad \tau_n \leq t < t_0.$$

By the evolution equation (3.2) it follows for $\tau_n \leq t < t_0$

$$\begin{aligned}\|f(\sigma, t) - f(\sigma, \tau_n)\| \leq & \int_{\tau_n}^{t} [\, \frac{\|f\|^8}{2}(|\Delta H| + |Q(A^0)H|)\,](\sigma, s)ds \\ \leq & c\,\|f(\sigma, \tau_n)\|^8 \int_{\tau_n}^{t_0} (|\nabla^2 A| + |A|^3)(\sigma, s)ds.\end{aligned}$$

Letting $t \nearrow t_0$ we conclude

$$\infty \longleftarrow \|f(\sigma, \tau_n)\|^{-7} \leq \int_{\tau_n}^{t_0} (|\nabla^2 A| + |A|^3)(\sigma, s)ds.$$

This shows in contradiction to (12.2)

$$\sup_{0\leq t<t_0}(\|\nabla^2 A\|_{L^\infty_\mu(\Sigma)} + \|A\|_{L^\infty_\mu(\Sigma)}) = \infty.$$

The additional estimates in theorem 12.2 follow from $T > t_0 = \min(\lambda, t_b)$ recalling the definitions of t_b and t_0 taking $k(n) := 3\Gamma(n)$. $\square$

Proof of theorem 1.1. Rescaling we may assume $\rho = 1$. Let $x \in \mathbb{R}^n$.
Using a suitable cut-off function and integration by parts one obtains

$$\int_{B_1(x)} |\nabla A|^2 d\mu \leq c \int_{B_2(x)} |A|^2 d\mu + c \int_{B_2(x)} |\nabla^2 A|^2 d\mu$$

for some constant $c = c(n) > 0$.
Covering $B_2(x) \subseteq \cup_{i \in \Gamma = \Gamma(n)} B_1(x_i)$ we obtain passing to the suprema

$$\sup_{x \in \mathbb{R}^n} \int_{B_1(x)} |\nabla A|^2 d\mu \leq c \sup_{x \in \mathbb{R}^n} \int_{B_1(x)} |A|^2 d\mu + c \sup_{x \in \mathbb{R}^n} \int_{B_1(x)} |\nabla^2 A|^2 d\mu.$$

Apply theorem 12.2 with $\alpha_2 = \alpha,\ \alpha_1 = c\delta + c\alpha_2,\ \rho = \frac{1}{2}$. □

Proof of theorem 1.2. The proof is, due to the stronger assumptions made, a simpler version of the one of theorem 12.2. Rescaling we may assume $\rho = 1$. Again we take $\delta := \frac{\varepsilon}{3\Gamma}$ and have for some $\Gamma = \Gamma(n) > 0$ and arbitrary $\lambda > 0$

$$\varepsilon(t) := \sup_{x \in \mathbb{R}^n} \int_{B_2(x)} |A|^2 d\mu \leq \Gamma \sup_{x \in \mathbb{R}^n} \int_{B_1(x)} |A|^2 d\mu,$$

$$t_0 := \sup\{0 \leq t < \min(T, \lambda) \mid \varepsilon(\tau) \leq 3\Gamma\delta \text{ for all } 0 \leq \tau < t\} > 0.$$

Taking $\lambda := \frac{c(n)}{R^8 + R^4}$ as in the proof of theorem 12.2 we derive analogously

$$t_0 = \min(T, \frac{c(n)}{R^8 + R^4}),$$

since by assumption

$$\sup_{0 \leq t < \min\{T, \frac{c(n)}{R^8 + R^4}\}} \|f\|_{L^\infty_\mu(\Sigma)} \leq R.$$

If $t_0 = \frac{c(n)}{R^8 + R^4}$, then $T > c(n) \frac{1}{R^8 + R^4}$, which is the required result.
If $t_0 = T$, then we have

$$t_0 \leq \frac{c(n)}{R^8 + R^4},$$

$$\sup_{0 \leq t < t_0, x \in \mathbb{R}^n} \int_{B_2(x)} |A|^2 d\mu \leq \varepsilon,$$

$$\sup_{0 \leq t < t_0} \|f\|_{L^\infty_\mu(\Sigma)} \leq 2R.$$

and the contradiction follows as in the case $t_0 = T$ in the proof of 12.2.
Clearly we may choose $k(n) = 3\Gamma(n)$.

□

13 Appendix

Lemma 13.1. *(Gronwall's inequality)*
Let $u, \alpha \in C^0([0,T),\mathbb{R})$ and $\beta \in C^0([0,T),\mathbb{R}_{\geq 0})$. Then we have

$$u(t) \leq \alpha(t) + \int_0^t \beta(s)u(s)\,ds \quad \Longrightarrow \quad u(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)e^{\int_s^t \beta(\sigma)\,d\sigma}ds.$$

Proof. Consider $y(t) := e^{-\int_0^t \beta(\sigma)d\sigma}\int_0^t \beta(s)u(s)\,ds$. By assumption

$$\begin{aligned} y'(t) =& -\beta(t)e^{-\int_0^t \beta(\sigma)d\sigma}\int_0^t \beta(s)u(s)\,ds + \beta(t)u(t)e^{-\int_0^t \beta(\sigma)d\sigma} \\ =& \Big[u(t) - \int_0^t \beta(s)u(s)\,ds\,\Big]\,\beta(t)e^{-\int_0^t \beta(\sigma)d\sigma} \leq \alpha(t)\beta(t)e^{-\int_0^t \beta(\sigma)d\sigma}. \end{aligned}$$

Hence by integration

$$e^{-\int_0^t \beta(\sigma)d\sigma}\int_0^t \beta(s)u(s)\,ds = y(t) - y(0) \leq \int_0^t \alpha(s)\beta(s)e^{-\int_0^s \beta(\sigma)d\sigma}ds,$$

what proves the claim by multiplicating both sides with $e^{\int_0^t \beta(\sigma)d\sigma}$. □

Let in the following $f : \Sigma \longrightarrow \mathbb{R}^n$ be a smooth immersion.

Lemma 13.2. *For $i \geq 1$ we have*

$$\begin{aligned} \nabla^i \|f\|^2 =& \sum_{2j+k=i-1} \langle f, \nabla f\rangle * P_{2j}^k(A) + \sum_{2n+m=i-2} \langle \nabla f, \nabla f\rangle * P_{2n}^m(A) \\ &+ \sum_{l+2q+p=i-2} \langle f, \nabla^l A\rangle * P_{2q}^p(A). \end{aligned}$$

Proof. The claim holds obviously true for $i = 1$. Inductively

$$\begin{aligned} \nabla^{i+1}\|f\|^2 =& \nabla\nabla^i\|f\|^2 \\ =& \sum_{2j+k=i-1} [\langle \nabla f, \nabla f\rangle * P_{2j}^k(A) + \langle f, A\rangle * P_{2j}^k(A) + \langle f, \nabla f\rangle * P_{2j}^{k+1}(A)] \\ &+ \sum_{2n+m=i-2} [\nabla\langle \nabla f, \nabla f\rangle * P_{2n}^m(A) + \langle \nabla f, \nabla f\rangle * P_{2n}^{m+1}(A)] \\ &+ \sum_{l+2q+p=i-2} [\langle \nabla f, \nabla^l A\rangle * P_{2q}^p(A) + \langle f, \partial^T\nabla^l A + \nabla^{l+1}A\rangle + P_{2q+1}^p \\ &\qquad + \langle f, \nabla^l A\rangle + P_{2q}^{p+1}(A)], \end{aligned}$$

where $\partial^2 f = A + \Gamma\partial f$ was used.
Clearly $\nabla\langle\nabla f, \nabla f\rangle = \nabla g = 0$ and $\langle\nabla f, \nabla^l A\rangle = 0$. Hence

$$\begin{aligned}\nabla^{i+1}\|f\|^2 = &\sum_{2j+k=i} \langle f, \nabla f\rangle * P^k_{2j}(A) + \sum_{2n+m=i-1} \langle\nabla f, \nabla f\rangle * P^m_{2n}(A) \\ &+ \sum_{l+2q+p=i-1} \langle f, \nabla^i A\rangle * P^p_{2q}(A) + \sum_{l+2q+p=i-2} \langle f, \partial^T\nabla^l A\rangle * P^p_{2q}(A).\end{aligned}$$

We have

$$\begin{aligned}\langle f, \partial^T_k\nabla^l A\rangle =& - \langle\partial_k f^T, \nabla^l A\rangle = -\langle\partial_k(g^{n,m}\langle f, \partial_n f\rangle\partial_m f), \nabla^l A\rangle \\ =& - g^{n,m}\langle f, \partial_n f\rangle\langle\partial_k\partial_m f, \nabla^l A\rangle = -g^{n,m}\langle f, \partial_n f\rangle\langle A_{k,m}, \nabla^l A\rangle \\ =&\langle f, \nabla f\rangle * P^l_2(A)\end{aligned}$$

and therefore

$$\begin{aligned}\sum_{l+2q+p=i-2} \langle f, \partial^T\nabla^l A\rangle * P^p_{2q}(A) =& \sum_{l+2q+p=i-2} \langle f, \nabla f\rangle * P^{p+l}_{2q+2}(A) \\ =& \sum_{2j+k=i} \langle f, \nabla f\rangle * P^k_{2j}(A).\end{aligned}$$

□

Let $P^l_k(|\nabla^\cdot A|)$ denote any term of the type $\Pi^k_{i=1}|\nabla^{l_i} A|$ with $\sum^k_{i=1} l_i = l$, where we freely use $k = 0 \Longrightarrow l = 0$ and define $P^0_0(|\nabla^\cdot A|) = 1$.

Corollary 13.3. *For $i \geq 1$ we have*

$$|\nabla^i\|f\|^2| \leq \|f\| \sum_{l+k=i-1} P^l_k(|\nabla^\cdot A|) + \sum_{2n+m=i-2} P^m_{2n}(|\nabla^\cdot A|).$$

Proof. Clearly $|\nabla f|^2 = g^{i,j}\langle\partial_i f, \partial_j f\rangle = g^{i,j}g_{i,j} = \delta^i_i = dim(\Sigma)$ and

$$\sum_{l+2q+p=i-2} \langle f, \nabla^l A\rangle * P^p_{2q}(A) = \sum_{l+2q+p=i-2} f * P^{p+l}_{2q+1}(A) = \sum_{l+k=i-1} f * P^l_k(A).$$

□

From this one obtains

Corollary 13.4. *We have for $i \geq 1$*

$$\begin{aligned}
|\nabla^i \|f\|^8| &\leq \sum_{m=1}^{8} \|f\|^{8-m} \sum_{l+k=i-m} P_k^l(|\nabla^{\cdot} A|) \\
&\leq c(i)\|f\|^7[|\nabla^{i-2}A| + |\nabla^{i-3}A||A| + |\nabla^{i-4}A|(|\nabla A| + |A|^2) + \ldots] \\
&\quad + \|f\|^6[\qquad |\nabla^{i-3}A| \quad + |\nabla^{i-4}A| \; |A| \qquad + \ldots] \\
&\quad + \|f\|^5[\qquad\qquad |\nabla^{i-4}A| \qquad\qquad + \ldots] \\
&\quad + \ \ldots \ ,
\end{aligned}$$

where we freely use $k = 0 \Longrightarrow l = 0$ and define $P_0^0(|\nabla^{\cdot} A|) = 1$.

We calculate the scaling behaviour of some geometric objects.

Lemma 13.5. *Let $f : \Sigma \longrightarrow \mathbb{R}^n$ be a smooth immersion, $d := dim(\Sigma)$.*

1. *$g_{\rho f} = \rho^2 g_f$ and $g^-_{\rho f} = \rho^{-2} g^-_f$ for the metric,*
2. *$\Gamma_{\rho f} = \Gamma_f$ for the Christoffel symbols,*
3. *$d\mu_{\rho f} = \rho^d d\mu_f$ for the area measure,*
4. *$|\nabla^m_{\rho f} A_{\rho f}|^2_{g_{\rho f}} = \rho^{-2(m+1)} |\nabla^m_f A_f|^2_{g_f}$,*
5. *$\sum_{(i,j,k) \in I(m),\, j<m+4} \nabla^i_{\rho f} \|\rho f\|^8 * P^j_k(A_{\rho f}) * \nabla^m_{\rho f} A_{\rho f}$*
   $= \rho^{-2(m-1)} \sum_{(i,j,k) \in I(m),\, j<m+4} \nabla^i_f \|f\|^8 * P^j_k(A_f) * \nabla^m_f A_f.$

Proof. (1) is clear, (2) follows from (1) and

$$\Gamma^k_{i,j} = \frac{1}{2} g^{k,m}(\partial_i g_{j,m} + \partial_j g_{i,m} - \partial_m g_{i,j}).$$

(3) follows from

$$J(\rho f) = \sqrt{det(d(\rho f)^T \circ d(\rho f))} = \sqrt{det(\rho^2 df^T df)} = \rho^d Jf.$$

To check (4) we derive $\nabla^m_{\rho f} A_{\rho f} = \rho \nabla^m_f A_f$ by $\nabla_{\rho f} = \nabla_f$, the scaling invariance of the covariant derivative and a simple induction argument. Hence

$$
\begin{aligned}
&|\nabla^m_{\rho f} A_{\rho f}|^2_{g_{\rho f}} \\
&= \sum_{i_1,\ldots,i_{m+2}\in\{1,2\}} \langle (\nabla^m_{\rho f} A_{\rho f})(e^{\rho f}_{i_1},\ldots,e^{\rho f}_{i_{m+2}}), (\nabla^m_{\rho f} A_{\rho f})(e^{\rho f}_{i_1},\ldots,e^{\rho f}_{i_{m+2}})\rangle \\
&= g^{i_1,i'_1}_{\rho f} \ldots g^{i_{m+2},i'_{m+2}}_{\rho f} \langle (\nabla^m_{\rho f} A_{\rho f})(\partial_{i_1},\ldots,\partial_{i_{m+2}}), (\nabla^m_{\rho f} A_{\rho f})(\partial_{i'_1},\ldots,\partial_{i'_{m+2}})\rangle \\
&= \rho^{-2(m+2)+2} g^{i_1,i'_1}_f \ldots g^{i_{m+2},i'_{m+2}}_f \\
&\qquad\qquad \langle (\nabla^m_f A_f)(\partial_{i_1},\ldots,\partial_{i_{m+1}}), (\nabla^m_f A_f)(\partial_{i'_1},\ldots,\partial_{i'_{m+1}})\rangle \\
&= \rho^{-2(m+1)} |\nabla^m_f A_f|^2_{g_f}.
\end{aligned}
$$

Likewise we obtain for $(i,j,k) \in I(m)$ recalling (4.6),

$$
\begin{aligned}
\nabla^i_{\rho f} \|\rho f\|^8 * P^j_k(A_{\rho f}) * \nabla^m_{\rho f} A_{\rho f} \\
&= \nabla^i_{\rho f} \|\rho f\|^8 * \nabla^{n_1}_{\rho f} A_{\rho f} * \ldots * \nabla^{n_k}_{\rho f} A_{\rho f} * \nabla^m_{\rho f} A_{\rho f} \\
&= \rho^{-i} \nabla^i_f \rho^8 \|f\|^8 * \rho^{-(n_1+1)} \nabla^{n_1}_f A_f * \ldots \\
&\qquad \ldots * \rho^{-(n_k+1)} \nabla^{n_k}_{\rho f} A_{\rho f} * \rho^{-(m+1)} \nabla^m_{\rho f} A_{\rho f} \\
&= \rho^{-i+8-(n_1+1)-\ldots-(n_k+1)-(m+1)} \nabla^i_f \|f\|^8 * P^j_k(A_f) * \nabla^m_f A_f \\
&= \rho^{-2(m-1)} \nabla^i_f \|f\|^8 * 2P^j_k(A_f) * \nabla^m_f A_f,
\end{aligned}
$$

since $n_1 + \ldots + n_k = j, \quad i+j+k = m+5$. This proves (5). ☐

The Willmore functional is invariant under inversion for closed surfaces.

Proposition 13.6. *Let $f \in C^2(\Sigma, \mathbb{R}^n \setminus \{\mathbb{0}\})$ be a closed immersed surface, $I : \mathbb{R}^n \setminus \{\mathbb{0}\} \longrightarrow \mathbb{R}^n \setminus \{\mathbb{0}\} : x \to \frac{x}{\|x\|^2}$. Then we have*

$$W(f) := \frac{1}{4}\int_\Sigma |H_f|^2 d\mu_f = \frac{1}{4}\int_\Sigma |H_{I_\sharp f}|^2 d\mu_{I_\sharp f} = W(I_\sharp f),$$

where $I_\sharp f(x) := I(f(x))$ for $x \in \Sigma$.

Proof. For the metric we have

$$
\begin{aligned}
g^{I_\sharp f}_{i,j} &= \langle \partial_i \frac{f}{\|f\|^2}, \partial_j \frac{f}{\|f\|^2}\rangle = \langle \frac{\partial_i f}{\|f\|^2} - 2\frac{\langle \partial_i f, f\rangle}{\|f\|^4} f, \frac{\partial_j f}{\|f\|^2} - 2\frac{\langle \partial_j f, f\rangle}{\|f\|^4} f\rangle \\
&= \frac{1}{\|f\|^4} g^f_{i,j},
\end{aligned}
$$

hence $g^{i,j}_{I_\sharp f} = \|f\|^4 g^{i,j}_f$. To calculate the mean curvature

$$H_{I_\sharp f} = g^{i,j}_{I_\sharp f} A^{I_\sharp f}_{i,j}, \quad A^{I_\sharp f}_{i,j} = \partial_i^{\perp_{I_\sharp f}} \partial_j I_\sharp f = \partial_i \partial_j I_\sharp f - P^{T_{I_\sharp f}}(\partial_i \partial_j I_\sharp f)$$

we derive

$$\begin{aligned}
\partial_i \partial_j I_\sharp f =& \partial_i \Big(\frac{\partial_j f}{\|f\|^2} - 2 \frac{\langle \partial_j f, f \rangle}{\|f\|^4} f \Big) \\
=& \frac{\partial_i \partial_j f}{\|f\|^2} - 2 \frac{\langle \partial_i f, f \rangle}{\|f\|^4} \partial_j f - 2 \frac{\langle \partial_j f, f \rangle}{\|f\|^4} \partial_i f \\
& - 2 \frac{\langle \partial_i \partial_j f, f \rangle}{\|f\|^4} f - 2 \frac{\langle \partial_i f, \partial_j f \rangle}{\|f\|^4} f + 8 \frac{\langle \partial_i f, f \rangle \langle \partial_j f, f \rangle}{\|f\|^6} f,
\end{aligned}$$

and

$$\begin{aligned}
P^{T_{I_\sharp f}}(\partial_i \partial_j I_\sharp f) =& g^{k,l}_{I_\sharp f} \langle \partial_i \partial_j I_\sharp f, \partial_k \frac{f}{\|f\|^2} \rangle \partial_l \frac{f}{\|f\|^2} \\
=& \|f\|^4 g^{k,l}_f \langle \partial_i \partial_j I_\sharp f, \frac{\partial_k f}{\|f\|^2} - 2 \frac{\langle \partial_k f, f \rangle}{\|f\|^4} f \rangle \Big(\frac{\partial_l f}{\|f\|^2} - 2 \frac{\langle \partial_l f, f \rangle}{\|f\|^4} f \Big) \\
=& g^{k,l}_f \langle \partial_i \partial_j I_\sharp f, \partial_k f \rangle \partial_l f \\
& - \frac{2}{\|f\|^2} g^{k,l}_f \langle \partial_i \partial_j I_\sharp f, \partial_k f \rangle \langle \partial_l f, f \rangle f \\
& - \frac{2}{\|f\|^2} g^{k,l}_f \langle \partial_i \partial_j I_\sharp f, f \rangle \langle \partial_k f, f \rangle \partial_l f \\
& + \frac{4}{\|f\|^4} g^{k,l}_f \langle \partial_i \partial_j I_\sharp f, f \rangle \langle \partial_k f, f \rangle \langle \partial_l f, f \rangle f.
\end{aligned}$$

For the several summands of the mean curvature we obtain

1.

$$g^{i,j}_{I_\sharp f} \partial_i \partial_j I_\sharp f = \|f\|^2 g^{i,j}_f \partial_i \partial_j f - 4 f^{T_f} - 2 \langle g^{i,j}_f \partial_i \partial_j f, f \rangle f - 4f + 8 \frac{\|f^{T_f}\|^2}{\|f\|^2} f,$$

2.

$$\begin{aligned}
& g^{i,j}_{I_\sharp f} g^{k,l}_f \langle \partial_i \partial_j I_\sharp f, \partial_k f \rangle \partial_l f \\
& \quad = P^{T_f} \Big(\|f\|^2 g^{i,j}_f \partial_i \partial_j f - 4 f^{T_f} - 2 \langle g^{i,j}_f \partial_i \partial_j f, f \rangle f - 4f + 8 \frac{\|f^{T_f}\|^2}{\|f\|^2} f \Big) \\
& \quad = \|f\|^2 g^{i,j}_f \partial_i^{T_f} \partial_j f - 8 f^{T_f} - 2 \langle g^{i,j}_f \partial_i \partial_j f, f \rangle f^{T_f} + 8 \frac{\|f^{T_f}\|^2}{\|f\|^2} f^{T_f},
\end{aligned}$$

3.

$$
\begin{aligned}
&-\frac{2}{\|f\|^2} g^{i,j}_{I_\sharp f} g^{k,l}_f \langle \partial_i \partial_j I_\sharp f, \partial_k f\rangle \langle \partial_l f, f\rangle f \\
&\quad = - \frac{2}{\|f\|^2}\langle \|f\|^2 g^{i,j}_f \partial_i\partial_j f - 4 f^{T_f} \\
&\qquad\qquad - 2\langle g^{i,j}_f \partial_i\partial_j f, f\rangle f - 4f + 8\frac{\|f^{T_f}\|^2}{\|f\|^2} f, f^{T_f}\rangle f \\
&\quad = - 2\langle g^{i,j}_f \partial_i\partial_j f, f^{T_f}\rangle f + 4\frac{\|f^{T_f}\|^2}{\|f\|^2}\langle g^{i,j}_f \partial_i\partial_j f, f\rangle f \\
&\qquad + 16\frac{\|f^{T_f}\|^2}{\|f\|^2} f - 16\frac{\|f^{T_f}\|^4}{\|f\|^4} f,
\end{aligned}
$$

4.

$$
\begin{aligned}
&-\frac{2}{\|f\|^2} g^{i,j}_{I_\sharp f} g^{k,l}_f \langle \partial_i \partial_j I_\sharp f, f\rangle \langle \partial_k f, f\rangle \partial_l f \\
&\quad = - \frac{2}{\|f\|^2}\langle \|f\|^2 g^{i,j}_f \partial_i\partial_j f - 4 f^{T_f} \\
&\qquad\qquad - 2\langle g^{i,j}_f \partial_i\partial_j f, f\rangle f - 4f + 8\frac{\|f^{T_f}\|^2}{\|f\|^2} f, f\rangle f^{T_f} \\
&\quad =2\langle g^{i,j}_f \partial_i\partial_j f, f\rangle f^{T_f} + 8\frac{\|f^{T_f}\|^2}{\|f\|^2} f^{T_f} + 8 f^{T_f} - 16\frac{\|f^{T_f}\|^2}{\|f\|^2} f^{T_f},
\end{aligned}
$$

5.

$$
\begin{aligned}
&\frac{4}{\|f\|^4} g^{i,j}_{I_\sharp f} g^{k,l}_f \langle \partial_i \partial_j I_\sharp f, f\rangle \langle \partial_k f, f\rangle \langle \partial_l f, f\rangle f \\
&\quad =\frac{4}{\|f\|^4}\langle \|f\|^2 g^{i,j}_f \partial_i\partial_j f - 4 f^{T_f} \\
&\qquad\qquad - 2\langle g^{i,j}_f \partial_i\partial_j f, f\rangle f - 4f + 8\frac{\|f^{T_f}\|^2}{\|f\|^2} f, f\rangle \|f^{T_f}\|^2 f \\
&\quad = - 4\frac{\|f^{T_f}\|^2}{\|f\|^2}\langle g^{i,j}_f \partial_i\partial_j f, f\rangle f + 16\frac{\|f^{T_f}\|^4}{\|f\|^4} f - 16\frac{\|f^{T_f}\|^2}{\|f\|^2} f.
\end{aligned}
$$

Collecting terms we conclude

$$
\begin{aligned}
H_{I_\sharp f} =& g^{i,j}_{I_\sharp f} A^{I_\sharp f}_{i,j} = g^{i,j}_{I_\sharp f}\partial_i\partial_j I_\sharp f - g^{i,j}_{I_\sharp f} P^{T_{I_\sharp f}}(\partial_i\partial_j I_\sharp f) \\
=& \|f\|^2 g^{i,j}_f \partial_i\partial_j f - 4f^{T_f} - 2\langle g^{i,j}_f\partial_i\partial_j f, f\rangle f - 4f + 8\frac{\|f^{T_f}\|^2}{\|f\|^2} f \\
& - (\|f\|^2 g^{i,j}_f \partial^{T_f}_i \partial_j f - 8f^{T_f} - 2\langle g^{i,j}_f \partial_i\partial_j f, f\rangle f^{T_f} + 8\frac{\|f^{T_f}\|^2}{\|f\|^2} f^{T_f}) \\
& - (-2\langle g^{i,j}_f\partial_i\partial_j f, f^{T_f}\rangle f + 4\frac{\|f^{T_f}\|^2}{\|f\|^2}\langle g^{i,j}_f\partial_i\partial_j f, f\rangle f \\
& \qquad + 16\frac{\|f^{T_f}\|^2}{\|f\|^2} f - 16\frac{\|f^{T_f}\|^4}{\|f\|^4} f) \\
& - (2\langle g^{i,j}_f\partial_i\partial_j f, f\rangle f^{T_f} + 8\frac{\|f^{T_f}\|^2}{\|f\|^2} f^{T_f} + 8f^{T_f} - 16\frac{\|f^{T_f}\|^2}{\|f\|^2} f^{T_f}) \\
& - (-4\frac{\|f^{T_f}\|^2}{\|f\|^2}\langle g^{i,j}_f\partial_i\partial_j f, f\rangle f + 16\frac{\|f^{T_f}\|^4}{\|f\|^4} f - 16\frac{\|f^{T_f}\|^2}{\|f\|^2} f) \\
=& \|f\|^2 H_f - 2\langle g^{i,j}_f\partial_i\partial_j f, f^{\perp_f}\rangle f - 4f^{T_f} - 4f + 8\frac{\|f^{T_f}\|^2}{\|f\|^2} f \\
=& \|f\|^2 H_f - 2\langle H_f, f\rangle f - 4f^{T_f} - 4f + 8\frac{\|f^{T_f}\|^2}{\|f\|^2} f.
\end{aligned}
$$

Hence we obtain for the squared absolute value

$$
\begin{aligned}
|H_{I_\sharp f}|^2 =& \|f\|^4 |H_f|^2 - 2\|f\|^2\langle H_f, f\rangle^2 - 4\|f\|^2\langle H_f, f\rangle + 8\|f^{T_f}\|^2\langle H_f, f\rangle \\
& - 2\|f\|^2\langle H_f, f\rangle^2 + 4\|f\|^2\langle H_f, f\rangle^2 + 8\|f^{T_f}\|^2\langle H_f, f\rangle \\
& + 8\|f\|^2\langle H_f, f\rangle - 16\|f^{T_f}\|^2\langle H_f, f\rangle \\
& + 8\|f^{T_f}\|^2\langle H_f, f\rangle + 16\|f^{T_f}\|^2 + 16\|f^{T_f}\|^2 - 32\frac{\|f^{T_f}\|^4}{\|f\|^2} \\
& - 4\|f\|^2\langle H_f, f\rangle + 8\|f\|^2\langle H_f, f\rangle + 16\|f^{T_f}\|^2 + 16\|f\|^2 - 32\|f^{T_f}\|^2 \\
& + 8\|f^{T_f}\|^2\langle H_f, f\rangle - 16\|f^{T_f}\|^2\langle H_f, f\rangle - 32\frac{\|f^{T_f}\|^4}{\|f\|^2} \\
& - 32\|f^{T_f}\|^2 + 64\frac{\|f^{T_f}\|^4}{\|f\|^2} \\
=& \|f\|^4|H_f|^2 + 8\|f\|^2\langle H_f, f\rangle - 16\|f^{T_f}\|^2 + 16\|f\|^2 \\
=& |\|f\|^2 H_f + 4f^{\perp_f}|^2.
\end{aligned}
$$

By (3.1) we have

$$
\begin{aligned}
\int_\Sigma |H_{I_\sharp f}|^2 d\mu_{I_\sharp f} &= \int_\Sigma \|f\|^{-4} | \, \|f\|^2 H_f + 4 f^{\perp_f}|^2 d\mu_f \\
&= \int_\Sigma |H_f|^2 d\mu_f + 8 \int_\Sigma \frac{\langle H_f, f\rangle}{\|f\|^2} d\mu_f + 16 \int_\Sigma \frac{|f^{\perp_f}|^2}{\|f\|^4} d\mu_f
\end{aligned}
$$

and

$$
\begin{aligned}
0 =& \int_\Sigma g_f^{i,j} \nabla_i (\|f\|^{-2} \langle \partial_j f, f\rangle) d\mu_f \\
=& -2 \int_\Sigma g_f^{i,j} \frac{\langle \partial_i f, f\rangle \langle \partial_j f, f\rangle}{\|f\|^4} d\mu_f + \int_\Sigma g_f^{i,j} \frac{\langle \partial_i \partial_j f, f\rangle}{\|f\|^2} d\mu_f \\
&+ \int_\Sigma g_f^{i,j} \frac{\langle \partial_i f, \partial_j f\rangle}{\|f\|^2} d\mu_f - \int_\Sigma g_f^{i,j} \Gamma_{i,j}^k \frac{\langle \partial_k f, f\rangle}{\|f\|^2} d\mu_f \\
=& -2 \int_\Sigma \frac{\|f^{T_f}\|^2}{\|f\|^4} d\mu_f + 2 \int_\Sigma \frac{1}{\|f\|^2} d\mu_f + \int_\Sigma g_f^{i,j} \frac{\langle \partial_i \partial_j f - \Gamma_{i,j}^k \partial_k f, f\rangle}{\|f\|^2} d\mu_f \\
=& 2 \int_\Sigma \frac{|f^{\perp_f}|^2}{\|f\|^4} d\mu_f + \int_\Sigma \frac{\langle H_f, f\rangle}{\|f\|^2} d\mu_f,
\end{aligned}
$$

which concludes the proof. □

Lemma 13.7. *Let $f : \Sigma \longrightarrow \mathbb{R}^n$ be a smooth immersion.*
Then we have

$$
(\Delta H + Q(A^0) H)_{\rho f} = \rho^{-3} (\Delta H + Q(A^0) H)_f .
$$

Proof. From lemma 13.5 we infer

$$
\begin{aligned}
H_{\rho f} &= A_{\rho f}(e_i^{\rho f}, e_i^{\rho f}) = g_{\rho f}^{i,j} A_{\rho f}(\partial_i, \partial_j) = \rho^{-2} g_f^{i,j} \rho A_f(\partial_i, \partial_j) \\
&= \rho^{-1} H_f .
\end{aligned}
$$

Therefore the Lapace-Beltrami of the mean curvature becomes

$$
\begin{aligned}
(\Delta H)_{\rho f} &= \rho^{-1} \Delta_{\rho f}(H_f) = \rho^{-1} g_{\rho f}^{i,j} (\nabla_{\partial_i}^{\rho f} \nabla_{\partial_j}^{\rho f} H_f - \nabla_{\nabla_{\partial_i}^{\rho f} \partial_j}^{\rho f} H_f) \\
&= \rho^{-3} g_f^{i,j} (\nabla_{\partial_i}^f \nabla_{\partial_j}^f H_f - \nabla_{\nabla_{\partial_i}^f \partial_j}^f H_f) \\
&= \rho^{-3} (\Delta H)_f .
\end{aligned}
$$

Next we derive for $Q(A^0)$, cf (2.13),

$$\begin{aligned}(Q(A^0)H)_{\rho f} &= A^0_{\rho f}(e_i^{\rho f}, e_j^{\rho f})\langle A^0_{\rho f}(e_i^{\rho f}, e_j^{\rho f}), H_{\rho f}\rangle \\ &= \rho^{-1} g_{\rho f}^{k,l} g_{\rho f}^{n,m} A^0_{\rho f}(\partial_k, \partial_n)\langle A^0_{\rho f}(\partial_l, \partial_m), H_f\rangle \\ &= \rho^{-5} g_f^{k,l} g_f^{n,m} A^0_{\rho f}(\partial_k, \partial_n)\langle A^0_{\rho f}(\partial_n, \partial_m), H_f\rangle.\end{aligned}$$

Since the tracefree part of the second fundamental form satisfies by (2.12)

$$\begin{aligned}A^0_{\rho f}(\partial_i, \partial_j) =& A_{\rho f}(\partial_i, \partial_j) - \frac{1}{2} g_{\rho f}(\partial_i, \partial_j) H_{\rho f} \\ =& \partial_i\partial_j(\rho f) -_{\rho f} \Gamma^k_{i,j}\partial_k(\rho f) - \frac{1}{2}\langle \partial_i(\rho f), \partial_j(\rho f)\rangle \rho^{-1} H_f \\ =& \rho(\partial_i\partial_j f -_f \Gamma^k_{i,j}\partial_k f - \frac{1}{2} g_f(\partial_i, \partial_j) H_f) \\ =& \rho A^0_f(\partial_i, \partial_j),\end{aligned}$$

we conclude

$$(Q(A^0)H)_{\rho f} = \rho^{-3} g_f^{k,l} g_f^{n,m} A^0_f(\partial_k, \partial_n)\langle A^0_f(\partial_l, \partial_m), H_f\rangle = \rho^{-3}(Q(A^0)H)_f.$$

□

We state two Sobolev-type inequalities.

Lemma 13.8. *Let $f : \Sigma \to \mathbb{R}^n$ be a smooth, immersed surface. Then we have for $u \in C^1_0(\Sigma)$*

$$(\int_\Sigma u^2 d\mu)^{\frac{1}{2}} \le c\,(\int_\Sigma |\nabla u| d\mu + \int_\Sigma |H||u| d\mu),$$

where $\mu = \mu_f$ and $c > 0$.

Proof. Cf. [2]. □

Lemma 13.9. *Let $f : \Sigma \to \mathbb{R}^n$ be a smooth, immersed surface. Furthermore*

$$2 < p \le \infty, 0 \le m \le \infty \text{ and } 0 < \alpha \le 1 \text{ with } \tfrac{1}{\alpha} = (\tfrac{1}{2} - \tfrac{1}{p})\, m + 1.$$

Then we have for $u \in C^1_0(\Sigma)$

$$\|u\|_{L^\infty_\mu(\Sigma)} \le c\|u\|^{1-\alpha}_{L^m_\mu(\Sigma)}(\|\nabla u\|_{L^p_\mu(\Sigma)} + \|Hu\|_{L^p_\mu(\Sigma)})^\alpha,$$

where $c = c(n, m, p) > 0$.

Proof. Cf. Theorem 5.6 in [1]. □

We prove some localized interpolation inequalities.
Let $f : \Sigma \to \mathbb{R}^n$ be a smooth immersion, $d = dim(\Sigma)$, Φ a l-linear form along f and $\gamma \in C_0^1(\Sigma)$ with $0 \leq \gamma \leq 1$, $\|\nabla\gamma\|_{L^\infty_\mu(\Sigma)} \leq \Lambda$.

Proposition 13.10. *Let* $1 \leq p, q, r < \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $\alpha + \beta = \eta + \theta = 1$. *For*

$$k \in \{0, \mathbb{R}_{\geq \eta p, \theta q}\},\ s \geq \alpha p, \beta q \text{ and } -\tfrac{s}{q} \leq u_1, u_2 \leq \tfrac{s}{p},\ -\tfrac{k}{q} \leq v_1, v_2 \leq \tfrac{k}{p}$$

we have

$$\begin{aligned}
&(\int_\Sigma \|f\|^k |\nabla\Phi|^{2r} \gamma^s d\mu)^{\frac{1}{r}} \\
&\qquad \leq ck(\int_{[\gamma>0]} \|f\|^{k-\eta p} |\Phi|^p \gamma^{s-u_1 p} d\mu)^{\frac{1}{p}} (\int_{[\gamma>0]} \|f\|^{k-\theta q} |\nabla\Phi|^q \gamma^{s+u_1 q} d\mu)^{\frac{1}{q}} \\
&\qquad\quad + c(\int_{[\gamma>0]} \|f\|^{k-v_2 p} |\Phi|^p \gamma^{s-u_2 p} d\mu)^{\frac{1}{p}} (\int_{[\gamma>0]} \|f\|^{k+v_2 q} |\nabla^2\Phi|^q \gamma^{s+u_2 q} d\mu)^{\frac{1}{q}} \\
&\qquad\quad + cs\Lambda(\int_{[\gamma>0]} \|f\|^{k-v_1 p} |\Phi|^p \gamma^{s-\alpha p} d\mu)^{\frac{1}{p}} (\int_{[\gamma>0]} \|f\|^{k+v_1 q} |\nabla\Phi|^q \gamma^{s-\beta q} d\mu)^{\frac{1}{q}},
\end{aligned}$$

where $0 \cdot \infty := 0$, $c = c(r, d)$ *and* $\|\nabla\gamma\|_{L^\infty_\mu(\Sigma)} \leq \Lambda$.

Proof. Let $k \geq \eta p, \theta q$. By integration by parts and Hölder's inequality

$$
\begin{aligned}
&\int_\Sigma \|f\|^k |\nabla\Phi|^{2r}\gamma^s d\mu \\
&\quad\leq ck\int_\Sigma \|f\|^{k-1}|\Phi||\nabla\Phi|^{2r-1}\gamma^s d\mu + c\int_\Sigma \|f\|^k|\Phi||\nabla\Phi|^{2r-2}|\nabla^2\Phi|\gamma^s d\mu \\
&\qquad + cs\Lambda\int_\Sigma \|f\|^k|\Phi||\nabla\Phi|^{2r-1}\gamma^{s-1}d\mu \\
&\quad= ck\int_\Sigma \|f\|^{\frac{k}{p}-\eta}|\Phi|\gamma^{\frac{s}{p}-u_1}\|f\|^{k\frac{r-1}{r}}|\nabla\Phi|^{2r-2}\gamma^{s\frac{r-1}{r}}\|f\|^{\frac{k}{q}-\theta}|\nabla\Phi|\gamma^{\frac{s}{q}+u_1}d\mu \\
&\qquad + c\int_\Sigma \|f\|^{\frac{k}{p}-v_2}|\Phi|\gamma^{\frac{s}{p}-u_2}\|f\|^{k\frac{r-1}{r}}|\nabla\Phi|^{2r-2}\gamma^{s\frac{r-1}{r}}\|f\|^{\frac{k}{q}+v_2}|\nabla^2\Phi|\gamma^{\frac{s}{q}+u_2}d\mu \\
&\qquad + cs\Lambda\int_\Sigma \|f\|^{\frac{k}{p}-v_1}|\Phi|\gamma^{\frac{s}{p}-\alpha}\|f\|^{k\frac{r-1}{r}}|\nabla\Phi|^{2r-2}\gamma^{s\frac{r-1}{r}}\|f\|^{\frac{k}{q}+v_1}|\nabla\Phi|\gamma^{\frac{s}{q}-\beta}d\mu \\
&\quad\leq ck\Big(\int_{[\gamma>0]} \|f\|^{k-\eta p}|\Phi|^p\gamma^{s-u_1p}d\mu\Big)^{\frac{1}{p}} \\
&\qquad\qquad \Big(\int_\Sigma \|f\|^k|\nabla\Phi|^{2r}\gamma^s d\mu\Big)^{\frac{r-1}{r}}\Big(\int_{[\gamma>0]} \|f\|^{k-\theta q}|\nabla\Phi|^q\gamma^{s+u_1q}d\mu\Big)^{\frac{1}{q}} \\
&\qquad + c\Big(\int_{[\gamma>0]} \|f\|^{k-v_2p}|\Phi|^p\gamma^{s-u_2p}d\mu\Big)^{\frac{1}{p}} \\
&\qquad\qquad \Big(\int_\Sigma \|f\|^k|\nabla\Phi|^{2r}\gamma^s d\mu\Big)^{\frac{r-1}{r}}\Big(\int_{[\gamma>0]} \|f\|^{k+v_2q}|\nabla^2\Phi|^q\gamma^{s+u_2q}d\mu\Big)^{\frac{1}{q}} \\
&\qquad + cs\Lambda\Big(\int_{[\gamma>0]} \|f\|^{k-v_1p}|\Phi|^p\gamma^{s-\alpha p}d\mu\Big)^{\frac{1}{p}} \\
&\qquad\qquad \Big(\int_\Sigma \|f\|^k|\nabla\Phi|^{2r}\gamma^s d\mu\Big)^{\frac{r-1}{r}}\Big(\int_{[\gamma>0]} \|f\|^{k+v_1q}|\nabla\Phi|^q\gamma^{s-\beta q}d\mu\Big)^{\frac{1}{q}}.
\end{aligned}
$$

The case $k = 0$ follows analogously. ☐

Corollary 13.11. *Let* $s \geq p \geq 2$, $k \in \{0, \mathbb{R}_{\geq p}\}$, $-\frac{k}{p} \leq u \leq \frac{k}{p}$, $-\frac{s}{p} \leq v \leq \frac{s}{p}$. *Then we have*

$$\begin{aligned}
&\int_\Sigma \|f\|^k |\nabla \Phi|^p \gamma^s d\mu \\
&\quad \leq \varepsilon \int_{[\gamma>0]} \|f\|^{k+up} |\nabla^2 \Phi|^p \gamma^{s+vp} d\mu \\
&\qquad + c(\varepsilon, d, p) \int_{[\gamma>0]} [\, k^p \|f\|^{k-p} \gamma^s + \|f\|^{k-up} \gamma^{s-vp} + s^p \Lambda^p \|f\|^k \gamma^{s-p}] |\Phi|^p d\mu.
\end{aligned}$$

Proof. Applying proposition 13.10 with

$$p = q = 2r,\ \alpha = \eta = 1,\ \beta = \theta = 0,\ u_1 = v_1 = 0$$

we obtain for $-\frac{k}{q} \leq u := v_2 \leq \frac{k}{p}$, $-\frac{s}{q} \leq v := u_2 \leq \frac{s}{p}$, since $s > 0$,

$$\begin{aligned}
&\Big(\int_\Sigma \|f\|^k |\nabla \Phi|^p \gamma^s d\mu\Big)^{\frac{2}{p}} \\
&\quad \leq ck\Big(\int_{[\gamma>0]} \|f\|^{k-p} |\Phi|^p \gamma^s d\mu\Big)^{\frac{1}{p}} \Big(\int_\Sigma \|f\|^k |\nabla \Phi|^p \gamma^s d\mu\Big)^{\frac{1}{p}} \\
&\qquad + c\Big(\int_{[\gamma>0]} \|f\|^{k-up} |\Phi|^p \gamma^{s-vp} d\mu\Big)^{\frac{1}{p}} \Big(\int_{[\gamma>0]} \|f\|^{k+up} |\nabla^2 \Phi|^p \gamma^{s+vp} d\mu\Big)^{\frac{1}{p}} \\
&\qquad + cs\Lambda\Big(\int_{[\gamma>0]} \|f\|^k |\Phi|^p \gamma^{s-p} d\mu\Big)^{\frac{1}{p}} \Big(\int_\Sigma \|f\|^k |\nabla \Phi|^p \gamma^s d\mu\Big)^{\frac{1}{p}}.
\end{aligned}$$

Monotonicity and convexity of $t \to t^{\frac{p}{2}}$, $t \geq 0$, and Young's inequality yield

$$\begin{aligned}
&\int_\Sigma \|f\|^k |\nabla \Phi|^p \gamma^s d\mu \\
&\quad \leq \varepsilon \int_\Sigma \|f\|^k |\nabla \Phi|^p \gamma^s d\mu + \varepsilon \int_{[\gamma>0]} \|f\|^{k+up} |\nabla^2 \Phi|^p \gamma^{s+vp} d\mu \\
&\qquad + c(\varepsilon, n, p) \int_{[\gamma>0]} [\, k^p \|f\|^{k-p} \gamma^s + \|f\|^{k-up} \gamma^{s-vp} + \Lambda^p \|f\|^k \gamma^{s-p}\,] |\Phi|^p d\mu.
\end{aligned}$$

Absorption and replacing $(1-\varepsilon)^{-1}\varepsilon$ by ε prove the claim. $\square$

References

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